BRIGHT-DARK SOLITONS OF THE TWO-COMPONENT NONLOCAL NONLINEAR SCHRÖDINGER EQUATIONS COUPLED TO BOUSSINESQ EQUATION

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Abstract. Motivated by the potential applications of multi-component nonlocal nonlinear Schrödinger (NLS) and NLS-type equations in nonlinear optics, the two-component nonlocal NLS equation coupled to the Boussinesq (2CNNLS-Boussinesq) equation is proposed and investigated. By employing the Kadomtsev-Petviashvili hierarchy reduction method, the multiple bright-dark soliton solutions, namely, one component featuring solitons with nonzero boundary condition and the other two components featuring solitons with zero boundary condition, are constructed in determinant forms. Based on the asymptotic analysis for the two-solitons, we bring out that the dark two-solitons possess three non-degenerate types and two degenerate types, while the bright two-solitons only admit one non-degenerate type. Additionally, we also consider the resonant-type collision of the bright-dark two-solitons, which is resulted by the phase shifts tending to zero in the collision process. The resonant-type collision can generate periodic waves in the region where the two solitons intersect. For the bright-dark four-solitons, we consider the bound state two-solitons pairs and the corresponding resonant-type collision between the two-solitons pairs. Finally, we also propose the arbitrary $N$ component nonlocal NLS-Boussinesq equation and some other nonlocal versions of the 2CNNLS-Boussinesq equations.

Key words: Two-component nonlocal nonlinear Schrödinger equations coupled to Boussinesq equation, Bright-dark solitons, Kadomtsev-Petviashvili hierarchy reduction method.

1. INTRODUCTION

In the past two decades, the study of parity-time ($PT$) symmetry has attracted extensive attentions in various physical settings because it can offer strong mathematical and physical insights to the underlying dynamical system [1–6]. In 2013, Ablowitz and Musslimani [7] first proposed the following nonlocal nonlinear Schrö-
dinger (NNLS) equation

\[ u_t(x,t) + u_{xx}(x,t) \pm 2u(x,t)V(x,t) = 0, \]
\[ V(x,t) = u(x,t)u^*(-x,t), \]

by employing a novel reduction of the well-known Ablowitz-Kaup-Newell-Segur (AKNS) system [8], where * stands for complex conjugation. In Eq. (1), the self-induced potential \( V \) satisfies the \( PT \)-symmetry condition \( V^*(-x,t) = V(x,t) \). By now, the nonlocal NLS equation (1) has been intensively studied from different points of view, including its integrability properties and solutions dynamics. Similar to the famous local NLS equation, the nonlocal NLS equation (1) also admits the Lax pair and an infinite number of conservation laws, and it was first solved by the inverse scattering transform method [8]. Additionally, typical localized wave structures of the nonlocal NLS equation (1) and their dynamical features have been constructed by different analytical methods [9–29], such as exact solitons, breathers, and rogue wave solutions. Here we have to note that the localized wave structures of nonlinear evolution equations have important applications in optics, matter-wave media, and in other physical settings [30–42].

As both of the local and nonlocal NLS equation can be derived from the AKNS system under different reductions, thus they admit a connection that can translate the local NLS equation to nonlocal NLS equation [29]. The nonlocal NLS equation (1) is gauge equivalent to an unconventional system of coupled Landau-Lifshitz equations [43].

Due to the potential applications of \( PT \)-symmetry in nonlinear optics and photonics, a hierarchy of nonlocal integrable NLS and NLS-type equations have proposed [44–48]. The nonlocal Davey-Stewartson (DS) equations and \( M \)-component nonlocal NLS equations are typical examples. The nonlocal DS equations provide integrable multidimensional versions of the nonlocal NLS equation (1), which were first introduced by Fokas [44] and Ablowitz and Musslimani [45]. The different types of rogue waves structures and other localized waveforms of nonlocal DS equations possessing various \( PT \)-symmetry conditions have been considered in [28, 49, 50]. The \( M \)-component nonlocal NLS equation [45–48] is a natural extension of the nonlocal NLS equation (1), and can be viewed as describing the propagation of multiple fields with \( u_\ell \) \((1 \leq \ell \leq M)\) being the complex envelope of the \( \ell \) optical field in a nonlocal nonlinear optical medium with a self-induced potential \( V(x,t) = \sum_{s=1}^{M} u_s(x,t)u_s^*(-x,t) \) that satisfies the \( PT \)-symmetry condition \( V^*(-x,t) = V(x,t) \) [51–53]. The bright solitons, energy-sharing collisions, and positons of the \( M \)-component nonlocal NLS equations have been considered in [54]. Some other localized wave structures were also reported in [55–60].

The two-component nonlocal NLS equation coupled to the Boussinesq equa-
tion is referred as two-component nonlocal NLS-Boussinesq (2CNNLS-Boussineq) equation,
\[ iu_{jt} + u_{j,xx} + Nu_j = 0, \quad j = 1, 2, \]
\[ \alpha (N_{xxxx} - 3(N^2)_{xx} - \beta N_{tt} + \gamma N_{xx} + \left( \sum_{s=1}^{2} \delta_s u_s u_s^*(−x, t) \right)_{xx} = 0, \quad (2) \]
which is another extension of the nonlocal NLS equation (1). If \( \alpha = 0, \beta = 0 \), the 2CNNLS-Boussinesq equation (2) reduces to the two-component nonlocal NLS equation. Equation (2) provides the PT-symmetric versions of the two-component local NLS-Boussinesq equation [61]. Motivated by the physical relevant applications of PT-symmetry, we consider the 2CNNLS-Boussinesq equation (2) with \( \alpha = 1, \beta = 3, \gamma = 1 \), namely, the following 2CNNLS-Boussinesq equation:
\[ iu_{jt} + u_{j,xx} + Nu_j = 0, \quad j = 1, 2, \]
\[ N_{xxxx} - 3(N^2)_{xx} - 3N_{tt} + N_{xx} + \left( \sum_{s=1}^{2} \delta_s u_s u_s^*(−x, t) \right)_{xx} = 0. \quad (3) \]
It is well known that the multi-component local NLS equation admits bright-dark solitons [62]. However, there are few researches of bright-dark solitons for the multi-component nonlocal systems. Thus, a natural motivation is to investigate the bright-dark solitons for the 2CNNLS-Boussinesq equation (3).

In this paper, general bright-dark solitons for the 2CNNLS-Boussinesq equation (3) are studied. The organization of this paper is as follows. In Sec. 2, we construct the general bright-dark solitons of the 2CNNLS-Boussinesq equation (3) by employing the KP-hierarchy reduction method. In Sec. 3, we investigate the dynamics of the collisions among \( 2M \)-solitons with zero and nonzero boundary conditions. The summary and discussion are given in Sec. 4.

2. BRIGHT-DARK SOLITON SOLUTIONS OF THE TWO-COMPONENT NONLOCAL NLS-BOUSSINESQ EQUATION

In this Section, we employ the KP hierarchy reduction method to obtain bright-dark soliton solutions for the nonlocal 2CNNLS-Boussinesq equation (3) with mixed boundary conditions, in which the solitons in the \( u_1 \) component are under nonzero boundary condition and in the \( u_2 \) component and \( N \) component are under zero boundary condition.

To use the tau functions of bilinear equations in KP hierarchy to construct solutions for the nonlocal 2CNNLS-Boussinesq equation (3), we have to cast the Eq. (3) into a set of bilinear equations, which is very close to a set of bilinear equations
of the KP hierarchy. We choose the following bilinear transformations to Eq. (3):

\[ u_1 = \sqrt{2g_f}, u_2 = \sqrt{2h_f}, N = (2\ln f)_{xx}, \]

and function \( f \) has to strictly meet the condition:

\[ f^*(-x, t) = f(x, t), \]

where \( f, g, \) and \( h \) are complex functions of variables \( x \) and \( t \). Then the Eq. (3) is casted into the following set of bilinear equations

\[
\begin{align*}
(D_x^2 + iD_t)g \cdot f &= 0, \\
(D_x^2 + iD_t)h \cdot f &= 0,
\end{align*}
\]

where \( D \) is the Hirota’s bilinear differential operator \[63\]. This set of bilinear equations is very close to the following set of bilinear equations in the multi-component KP hierarchy:

\[
\begin{align*}
(D_{x_1}^2 - D_{x_2})\tau_0^{(n+1)} \cdot \tau_0^{(n)} &= 0, \\
(D_{x_1}^2 - D_{x_2})\tau_1^{(n)} \cdot \tau_0^{(n)} &= 0, \\
D_{x_1}D_{y}\tau_0^{(n)} \cdot \tau_0^{(n)} &= -2\tau_1^{(n)} \tau_0^{(n)},
\end{align*}
\]

Referring to the Sato theory \[64–66\], the above set of bilinear equations in Eq. (7) has the following tau functions:

\[
\begin{align*}
\tau_0^{(n)} &= |M|, \\
\tau_1^{(n)} &= |M\Phi^T|, \\
\tau_{-1}^{(n)} &= |M\Psi^T|,
\end{align*}
\]

where the elements of matrix \( M \) are

\[
\begin{align*}
m_{s,j}^{(n)} &= \left(-\frac{p_s}{p_j}\right)^n \frac{e^{\xi_s + \bar{\xi}_j}}{p_s + p_j} + \frac{e^{\eta_s + \bar{\eta}_j}}{q_s + q_j},
\end{align*}
\]

with

\[
\begin{align*}
\xi_s &= \frac{1}{p_s} x_{-1} + p_s x_1 + p_s^2 x_2 + p_s^3 x_3, \\
\bar{\xi}_j &= \frac{1}{p_j} x_{-1} + p_j x_1 - p_j^2 x_2 + p_j^3 x_3,
\end{align*}
\]

for \( 1 \leq s, j \leq \hat{M} \), and the superscript \( T \) indicates transpose, \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \) are row vec-
tors defined by
\[ \Phi = (e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_M}), \Psi = (e^{\eta_1}, e^{\eta_2}, \ldots, e^{\eta_M}), \]
\[ \overline{\Phi} = (e^{\overline{\xi}_1}, e^{\overline{\xi}_2}, \ldots, e^{\overline{\xi}_M}), \overline{\Psi} = (e^{\overline{\eta}_1}, e^{\overline{\eta}_2}, \ldots, e^{\overline{\eta}_M}). \] (11)

We use the tau functions (8) to derive solutions of the bilinear equations of the nonlocal 2CNNLS-Boussinesq equation (6) by producing the following two steps.

The first step is to constrain the tau function being subject to the dimension reduction:
\[ (\partial_{x_1} - \delta_1 \partial_{x_{-1}} + \delta_2 \partial_{y} + 4 \partial_{x_3}) \tau_0^{(n)} = c \tau_0^{(n)}, \] (12)
which can be achieved by constraining the matrix elements obeying the relation:
\[ (\partial_{x_1} - \delta_1 \partial_{x_{-1}} + \delta_2 \partial_{y} + 4 \partial_{x_3}) \tau_0^{(n)} = 0. \] (13)
From Eq.(13) we can obtain the parameters \( p_s, q_j, \tilde{p}_s, \tilde{q}_j \) admitting the relations:
\[ q_j = 4p^4_s + p^2_s + \delta_1, \tilde{q}_s = \frac{4p^4_s + p^2_s + \delta_1}{\delta_2 p_s}. \] (14)
From the last three bilinear equations in Eq. (7) we yield the tau function \( \tau_0^{(n)} \) satisfying the bilinear equation:
\[ (D^4_{x_1} + D^2_{x_1} - 2 \delta_1) \tau_0^{(n)} \cdot \tau_0^{(n)} = -2 \delta_1 \tau_0^{(n+1)} \tau_0^{(n)} + 2 \delta_2 \tau_1^{(n)} \tau_{-1}^{(n)}. \] (15)
Under the dimension reduction (12), then the bilinear equation (15) reduces to
\[ (D^4_{x_1} + D^2_{x_1} + 3D^2_{x_2} - 2 \delta_1) \tau_0^{(n)} \cdot \tau_0^{(n)} = -2 \delta_1 \tau_0^{(n+1)} \tau_0^{(n)} + 2 \delta_2 \tau_1^{(n)} \tau_{-1}^{(n)}. \] (16)
This bilinear equation and the first two bilinear equations in Eq. (7) are only related to variables \( x_1, x_2 \), and the variables \( y, x_{-1}, x_3 \) vanish. By summing up the above results, the tau functions (8), with variables transformations \( x_1 = x, x_2 = i \ell, x_{-1} = 0, x_3 = 0, y = 0 \) and holding the parametric condition (14), satisfy the following set of 1+1-dimensional bilinear equations:
\[ (D^2_x + iD_t) \tau_0^{(n+1)} \cdot \tau_0^{(n)} = 0, \]
\[ (D^2_x + iD_t) \tau_1^{(n)} \cdot \tau_0^{(n)} = 0, \]
\[ (D^2_x + D^2_x + 3D^2_x - 2 \delta_1) \tau_0^{(n)} \cdot \tau_0^{(n)} = -2 \delta_1 \tau_0^{(n+1)} \tau_0^{(n)} + 2 \delta_2 \tau_1^{(n)} \tau_{-1}^{(n)}. \] (17)
This set of bilinear equations reduces to the bilinear equations (6) of the 2CNNLS-Boussinesq equation for
\[ \tau_0^{(0)} = f, \tau_0^{(1)} = g, \tau_1^{(0)} = h, \tau_0^{(-1)} = g^*(-x, t), \tau_{-1}^{(0)} = -h^*(-x, t), \]
if the tau functions satisfy the symmetry and complex conjugation:
\[ \tau_0^{(-n)}(-x, t) = \tau_0^{(n)}(x, t), \tau_1^{(-n)}(-x, t) = \tau_1^{(n)}(x, t). \] (18)
The second step is to constrain the tau functions $\tau_0^{(n)}, \tau_1^{(n)}, \tau_{-1}^{(n)}$ being subject to the above symmetry and complex conjugation (18). With variables transformations $x_1 = x, x_2 = it, x_{-1} = 0, x_3 = 0, y = 0$ and considering $\tilde{M} = 2M$ and the parametric constraints:

$$p_j = p_j^*, q_j = q_j^*, \tau_j = \eta_j, \quad j = 1, 2, \cdots 2M,$$

(19)

and

$$p_{N+s} = -p_s, q_{N+s} = -q_s, \eta_{N+s,0} = \eta_{s,0}^*, \quad s = 1, 2, \cdots M,$$

(20)

the matrix elements in Eq. (8) admit the relations

$$m_{N+s,N+j}^{(-n)*}(-x,t) = -m_{j,s}^{(-n)}(x,t), \quad m_{N+s,j}^{(-n)*}(-x,t) = -m_{N+j,s}^{(-n)}(x,t),$$

$$m_{s,N+j}^{(-n)*}(-x,t) = -m_{j,N+s}^{(-n)}(x,t), \quad m_{s,j}^{(-n)*}(-x,t) = -m_{N+j,N+s}^{(-n)}(x,t),$$

(21)

which can further yield

$$\tau_0^{(-n)*}(-x,t) = \left| \begin{array}{c} m_{s,j}^{(-n)}(-x,t) \\ \tilde{m}_{N+s,N+j}^{(-n)*}(-x,t) \\ m_{j,N+s,j}^{(-n)*}(-x,t) \\ \tilde{m}_{N+s,j}^{(-n)*}(-x,t) \end{array} \right| = (-1)^{2N} \tau_0^{(n)}(x,t),$$

(22)

thus $\tau_0^{(-n)*}(-x,t) = \tau_0^{(n)}(x,t)$. Similarly, the symmetry and complex conjugation of the tau function $\tau_1^{(-n)}$ (i.e., $\tau_1^{(-n)*}(-x,t)$) and the tau function $\tau_{-1}^{(n)}$ also hold:

$$\tau_1^{(-n)*}(-x,t) = \tau_{-1}^{(n)}(x,t).$$

(23)

Hence, the symmetry and complex conjugation in Eq. (18) is very well satisfied.

By summarizing the above results, and further taking $\eta_{s,0} = \tilde{\eta}_s$ for simplicity, the soliton solutions for the nonlocal 2CNNLS-Boussinesq equation are expressed by the following Theorem.

**Theorem 1.** The nonlocal 2CNNLS-Boussinesq equation (3) has the following general bright-dark soliton solutions

$$u_1 = \sqrt{2} \frac{g}{f}, \quad u_2 = \sqrt{2} \frac{h}{f}, \quad N = (2lnf)_{xx},$$

(24)
where
\[ f(x,t) = \begin{vmatrix} m_{s,j}^{(0)} & m_{s,N+j}^{(0)} \\ m_{N+s,j}^{(0)} & m_{N+s,N+j}^{(0)} \end{vmatrix}, \quad g(x,t) = \begin{vmatrix} m_{s,j}^{(1)} & m_{s,N+j}^{(1)} \\ m_{N+s,j}^{(1)} & m_{N+s,N+j}^{(1)} \end{vmatrix}, \]
\[
 h(x,t) = \begin{vmatrix} m_{s,j}^{(0)} & m_{s,N+j}^{(0)} e^{p_s x + ip_s^2 t} \\ m_{N+s,j}^{(0)} & m_{N+s,N+j}^{(0)} e^{-p_s x + ip_s^2 t} \\ -e^{\eta_s} & -e^{-\eta_s} \\ 0 \end{vmatrix},
\]
and
\[
 m_{s,j}^{(n)} = \frac{1}{p_s + p_j^*} (-p_s p_j^*)^{n/2} e^{\xi_s + \xi_j^*} + \frac{e^{\eta_s - \eta_j^*}}{q_s + q_j^*},
\]
\[ \xi_s = p_s x + ip_s^2 t, \quad q_s = \frac{4p_s^4 + p_s^2 + \delta_1}{\delta_2 p_s}, \]
for \( s, j = 1, 2, \cdots, M \). The complex parameters \( p_s, \eta_s \) have to obey the parametric constraints:
\[ p_{K+s} = -p_s, \eta_{K+s} = \eta_s^*. \]

Remark 1. The \( u_1 \) component corresponds to solitons with nonzero boundary condition, and the \( u_2 \) component and the \( N \) component are solitons with zero boundary condition.

3. DYNAMICS OF THE COLLISIONS OF 2N-SOLITONS WITH ZERO AND NONZERO BOUNDARY CONDITIONS

The solutions in Theorem 1 are 2N-soliton solutions with nonzero boundary condition (for the \( u_1 \) component) and zero boundary condition (for \( u_2 \) component) to the two-component nonlocal NLS-Boussinesq equation. In this section, we would study the collision behaviours of these 2N-solitons with different boundary conditions.

3.1. COLLISIONS OF BRIGHT-DARK TWO-SOLITONS

The bright-dark two-solitons in the two-component nonlocal NLS-Boussinesq equation (3) are derived from Theorem 1 with \( M = 1 \), and the functions \( f, g \) and \( h \) of the two-soliton solution (4) are written explicitly as the following determinant forms:

\[
 f = \begin{vmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} \\ m_{2,1}^{(0)} & m_{2,2}^{(0)} \end{vmatrix}, \quad g = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,2}^{(1)} \\ m_{2,1}^{(1)} & m_{2,2}^{(1)} \end{vmatrix}, \quad h = \begin{vmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} e^{p_1 x + ip_1^2 t} \\ m_{2,1}^{(0)} & m_{2,2}^{(0)} e^{-p_1 x + ip_1^2 t} \\ -e^{\eta_1} & -e^{-\eta_1} \\ 0 \end{vmatrix},
\]

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following asymptotic forms for the bright-dark two solitons: 

where 

\[ n^{(1)}_{1,1} = \frac{1}{p_1 + p_1'} \left( -\frac{p_1}{p_1'} \right) e^{\left( q_1 + q_1'\right)x + i\left( p_1 - p_1'\right) t - p_1'^2 t} + \frac{e^{2\eta_1}}{q_1 + q_1'}, \]

\[ n^{(1)}_{1,2} = \frac{1}{p_1 - p_1'} \left( \frac{p_1}{p_1'} \right) e^{\left( q_1 - q_1'\right)x + i\left( p_1 - p_1'\right) t + p_1'^2 t} + \frac{e^{2\eta_1}}{q_1 - q_1'}, \]

\[ n^{(1)}_{2,1} = -\frac{1}{p_1 - p_1'} \left( \frac{p_1}{p_1'} \right) e^{\left( q_1 - q_1'\right)x - i\left( p_1 - p_1'\right) t - p_1'^2 t} - \frac{e^{2\eta_1}}{q_1 - q_1'}, \]

\[ n^{(1)}_{2,2} = -\frac{1}{p_1 + p_1'} \left( \frac{p_1}{p_1'} \right) e^{\left( q_1 + q_1'\right)x - i\left( p_1 + p_1'\right) t - p_1'^2 t} - \frac{e^{2\eta_1}}{q_1 + q_1'}, \]

\[ q_1 = \frac{4p_1'^2 + p_1^2 + \delta_1}{\delta_2 p_1}. \]

This two-soliton solution comprises of two complex soliton parameters \( p_1, \eta_1 \), and two nonlinear coefficients \( \delta_1, \delta_2 \) related to the original equation (3). If the real part or the imaginary part of function \( f \) (i.e., the denominator of this two-soliton solution) is nonzero, the two-soliton solution will be non-singular. The imaginary part of function \( f \) can be explicitly written as the form:

\[ f_I = \frac{\eta_1 e^{-4p_1 R R t}}{4p_1 R q_1 R} \left( e^{2p_1 R x} - e^{-2p_1 R x} \right), \]

where \( \eta_1 = \text{Im}(e^{2\eta_1}) = \sin(2\eta_1 t) e^{2\eta_1 R}. \) Here and in the following context, the subscripts \( R \) and \( I \) stand for the real and imaginary parts of a given parameter or function, respectively. Note that \( p_1 R \neq 0 \) and \( f_I \neq 0 \) if \( \eta_1 \neq 0 \). Hereafter, we constrain \( \eta_1 I \neq \frac{n \pi}{2} \) in order to remain non-singular the function of the two-soliton solution (28).

To clearly understand the collision behaviors of these bright-dark two-soliton solutions, we have to do the asymptotic analysis for these bright-dark two-soliton. These two solitons move along \( \zeta_1 = x - 2p_1 R t \) and \( \zeta_2 = x + 2p_1 R t \), which are denoted as soliton 1 and soliton 2 in the following asymptotic analysis, respectively. We assume \( p_1 R > 0, p_1 I > 0 \) and can further yield: along \( \zeta_1 + \zeta_1' \approx 0, \zeta_2 + \zeta_2' \rightarrow \pm\infty \) as \( t \rightarrow \mp\infty \); along \( \zeta_2 + \zeta_2' \approx 0, \zeta_1 + \zeta_1' \rightarrow \mp\infty \) as \( t \rightarrow \mp\infty \). Then one can obtain the following asymptotic forms for the bright-dark two solitons:

(a) Before collision \( (t \rightarrow -\infty) \)

Soliton 1 \( (\zeta_1 \approx 0, \zeta_2 \rightarrow -\infty) \):

\[ u_1^{(1)} \approx z_1 \left[ 1 + z_1 + (z_1 - 1) \tanh(p_1 R \zeta_1 + \lambda_1 - \eta_1) \right]; \]

\[ u_2^{(1)} \approx A_1 e^{-i\eta_1} \text{sech}(p_1 R \zeta_1 + \lambda_1 - \eta_1); \]

\[ N^{(1)} \approx 2p_1 R \text{sech}^2(p_1 R \zeta_1 + \lambda_1 - \eta_1). \]
After algebraic calculations, the intensities of the two solitons in (i) Soliton patterns related to soliton intensities.

For components before collision are:

\begin{equation}
|u_1^{(1)}| = \sqrt{2} \left| \frac{1 + e^{2i(\tilde{\phi} - \tilde{\eta}I)}}{1 + e^{-2i\eta I}} \right|, |u_1^{(2)}| = \sqrt{2} \left| \frac{1 + e^{2i(\tilde{\phi} + \tilde{\eta}I)}}{1 + e^{2i\eta I}} \right|,
\end{equation}

\begin{equation}
|u_2^{(j)}| = \frac{\sqrt{p_{1R}q_{1I}}}{\cos \tilde{\eta}I} |\eta|, |N^{(j)}| = \frac{2p_{1R}^2}{\cos^2 \tilde{\eta}I}, j = 1, 2,
\end{equation}

Based on the asymptotic analysis, we can study the collision dynamics of the bright-dark two-soliton from the following three aspects:

**Soliton patterns related to soliton intensities.**

After algebraic calculations, the intensities of the two solitons in \( u_1, u_2 \), and \( N \) components before collision are:

\begin{align*}
|u_1^{(1)}| &= \sqrt{2} \left| \frac{1 + e^{2i(\tilde{\phi} - \tilde{\eta}I)}}{1 + e^{-2i\eta I}} \right|, |u_1^{(2)}| &= \sqrt{2} \left| \frac{1 + e^{2i(\tilde{\phi} + \tilde{\eta}I)}}{1 + e^{2i\eta I}} \right|, \\
|u_2^{(j)}| &= \frac{\sqrt{p_{1R}q_{1I}}}{\cos \tilde{\eta}I} |\eta|, |N^{(j)}| &= \frac{2p_{1R}^2}{\cos^2 \tilde{\eta}I}, j = 1, 2,
\end{align*}
and their intensities after collision are
\[
|u_1^{(1)+}| = \sqrt{2} \left| \frac{1 + e^{2i(\bar{\psi} - \bar{\eta}_I)}}{1 + e^{-2i\bar{\eta}_I}} \right|, |u_1^{(2)+}| = \sqrt{2} \left| \frac{1 + e^{2i(\bar{\psi} + \bar{\eta}_I)}}{1 + e^{2i\bar{\eta}_I}} \right|, \\
|u_2^{(j)+}| = \sqrt{\frac{p_{1R}^2 \bar{\eta}_I}{\cos^2 \bar{\eta}_I}}, |N^{(j)+}| = \frac{2p_{1R}^2}{\cos^2 \bar{\eta}_I},
\]
(37)
where \(z_1 = e^{2i\bar{\psi}}\). Since \(|u_j^{(\ell)+}| = |u_j^{(\ell)-}|, |N^{(\ell)+}| = |N^{(\ell)-}| (j, \ell = 1, 2)\), thus we can obtain that the two solitons have their intensities unaltered in the collision process. In what follows, we classify the two-solitons depending on their intensities for \(u_1, u_2\), and \(N\) components.

The \(u_1\) component features two-solitons on a background whose amplitude value is \(\sqrt{2}\), the soliton \(j (j = 1, 2)\) in the \(u_1\) component can be classified into three types: a dark soliton when \(\left| \frac{1 + e^{2i(\bar{\psi} - \bar{\eta}_I)}}{1 + e^{-2i\bar{\eta}_I}} \right| < 1\), an anti-dark soliton when \(\left| \frac{1 + e^{2i(\bar{\psi} - \bar{\eta}_I)}}{1 + e^{-2i\bar{\eta}_I}} \right| > 1\), and a soliton decaying into the background when \(\left| \frac{1 + e^{2i(\bar{\psi} - (-1)^j \bar{\eta}_I)}}{1 + e^{-2i\bar{\eta}_I}} \right| = 1\). Furthermore, the following two equations can not be satisfied for all the choices of \(\bar{\psi}\) and \(\bar{\eta}_I\)
\[
\left| \frac{1 + e^{2i(\bar{\psi} - \bar{\eta}_I)}}{1 + e^{2i\bar{\eta}_I}} \right| = 1, \left| \frac{1 + e^{2i(\bar{\psi} + \bar{\eta}_I)}}{1 + e^{2i\bar{\eta}_I}} \right| = 1,
\]
(38)
thus the two solitons can not decay into the background simultaneously. By summarizing the above analyses, the two solitons in the \(u_1\) component are sorted into three types of non-degenerated two-solitons (namely, anti-dark–anti-dark two-solitons, anti-dark–dark two-solitons, dark–dark two-solitons), and two types of degenerated two-solitons (i.e., anti-degenerated two-solitons, and dark degenerated two-solitons).

Figure 1 displays these three types of non-degenerated two-solitons: the two solitons feature anti-dark shape in Fig. 1(a), while they illustrate dark shape in Fig. 1(b) and a mixture of dark shape and anti-dark shape in Fig. 1(c). The two patterns of degenerated two-solitons, namely, the degenerated anti-dark two-solitons and the degenerated dark two-solitons, are demonstrated in Fig. 2(a) and Fig. 2(b), respectively.

The \(u_2\) and \(N\) components feature two solitons on zero boundary condition, Eqs. (36), (37) imply \(|u_2^{(1)\pm}| = |u_2^{(2)\pm}| > 0, |N^{(1)\pm}| = |N^{(2)\pm}| > 0\), namely, the intensities of the two solitons in \(u_2\) and \(N\) components remain unchanged and are higher than the amplitude of the zero background, thus the two-solitons in \(u_2\) and \(N\) components only possess bright waveforms. Additionally, the degenerated two-solitons do not exist in the \(u_2, N\) components. Figure 3 clearly shows these features. The velocities of the two solitons are \(\pm 2p_{1I}\), the signs \(\pm\) represent the directions of the two solitons movement. From Eqs. (36), (37) one can further derive that the intensities of the two solitons in the \(N\) component are independent of their velocities,
Fig. 1 – (Colour online) Four types of non-degenerated two-solitons in the $u_1$ component of the 2NLS-Boussinesq equation: (a) Anti-dark–anti-dark two-solitons with parameters $p_1 = 1 + i, \tilde{\eta}_1 = \frac{i\pi}{10}$; (b) Dark–Dark two-solitons with parameters $p_1 = 2 + i, \tilde{\eta}_1 = 0$; (c) Anti-dark–Dark two-solitons with parameters $p_1 = 1 + i, \tilde{\eta}_1 = \frac{i\pi}{4}$.

while they are dependent in the $u_2$ component. To study the relations between the intensities and the velocities of the two solitons in the $u_2$ component, we take $p_{1I} = \frac{v}{2}$, where $v$ is the velocity of the two solitons, then we can explicitly write down:

$$\hat{A} = \frac{p_{1R}}{\cos \tilde{\eta}_{1I}} \sqrt{\frac{2}{\delta_2} \left[ 4p_{1R}^2 - v^2 + 1 - \frac{4\delta_1}{4p_{1R}^2 + v^2} \right]}, \quad (39)$$

where $\hat{A}$ denotes the intensity of the two solitons in the $u_2$ component. As inferred from Eq. (39), $\hat{A}$ could be regarded as a function decreasing with respect to $v^2$ when $\delta_1 < 0$, which indicates that the two solitons possess higher intensities when they move at lower speeds. When $\delta_1 > 0$, the value of $\hat{A}$ could also be strongly affected by the value of $\delta_1$. These interesting features imply that the nonlinear coefficient
of the nonzero boundary condition component (i.e., \( \delta_1 \)) strongly influences the relations of intensities and velocities of the two solitons in the zero boundary condition component. Similarly, we have also examined that the nonlinear coefficient of the zero boundary condition component \( u_2 \) (i.e., \( \delta_2 \)) can also strongly affect the relations between the intensities and velocities of the two solitons in the \( u_1 \) component with nonzero boundary condition.

![Image](image.jpg)

**Fig. 3** – (Colour online) The intensity profiles of bright two-soliton along \( t = -2 \) (blue solid line) and \( t = 2 \) (red dashed line) in the \( u_2 \) and \( N \) components of the 2NLS-Boussinesq equation with \( p_1 = -1 + i, \tilde{\eta}_1 = \frac{i\pi}{4} \).

(ii) **Phase shifts.** The soliton phase being altered in the collision process is a striking feature, which is related to the resonant or resonant-type collision. It is observed that the initial phase and final phase of the soliton 1 and soliton 2 in the \( u_1 \) component with nonzero boundary condition are:

\[
\begin{align*}
\Phi_{1(1)}^- &= \lambda_1 - \tilde{\eta}_1, & \Phi_{1(1)}^+ &= \hat{\lambda}_1 - \tilde{\eta}_1, \\
\Phi_{1(2)}^- &= -\lambda_1 + \tilde{\eta}_1^*, & \Phi_{1(2)}^+ &= -\hat{\lambda}_1 + \tilde{\eta}_1^*. \\
\end{align*}
\]

(40)

where \( \Phi_{1(j)}^- \) and \( \Phi_{1(j)}^+ \) stand for the initial phase and final phase of the soliton \( j (= 1, 2) \). The final phase minus the initial phase can yield the phase shift of the soliton \( j \) during the collision process:

\[
\Delta \Phi_{1(1)} = -\Delta \Phi_{1(2)} = \hat{\lambda}_1 - \lambda_1 = \ln \left( \frac{p_{1R}^2 p_{1I}^2 q_{1I}^2}{q_{1R}^2 (p_{1R}^2 + p_{1I}^2)(q_{1R}^2 + q_{1I}^2)} \right),
\]

(41)

where \( q_1 \) is given by Eq. (26). Similarly, we also obtain the phase shift of the soliton
of the $u_2$ and $N$ components.

\[
\Delta \Phi_2^{(1)} = -\Delta \Phi_2^{(2)} = \lambda_1 - \lambda_1 = \sqrt{\frac{p_1^2 R \phi_2^2(q_1^2 I + q_1^2 I)}{q_1^2 R(q_1^2 R + p_1 I)}},
\]
\[
\Delta \Phi_N^{(1)} = -\Delta \Phi_N^{(2)} = -(\lambda_1 + \lambda_1) = \sqrt{\frac{q_2^2 R(p_1^2 R + p_1^2 I)}{p_1^2 I(q_1^2 R + q_1^2 I)}},
\]  

(42)

From the above expressions of $\Phi_{\ell}^{(j)}(\ell, j = 1, 2)$, the phase shifts are determined by the three parameters $p_1, \delta_1, \delta_2$. Additionally, for some local systems[67–72], the phase shift tending to infinity would lead to the resonant or resonant-type collisions. However, we found that the resonant collision or resonant-type collision would not happen when the phase shift tends to infinity. But the resonant-type collision occurs when the phase shifts tend to zero. Figure 4 shows such resonant collisions with parameters

\[
\delta_1 = 1, \delta_2 = 1, p_1 = \frac{1}{10} + i, \eta_1 = -\frac{i \pi}{4}.
\]  

(43)

With these parameters in the phase shift expressions given by Eqs. (41), (42), we can calculate that the phase shifts are $\Delta \Phi_1^{(1)} = -\Delta \Phi_1^{(2)} = 0.0703, \Delta \Phi_1^{(1)} = -\Delta \Phi_1^{(2)} = -0.085$. As can be seen in Fig. 4, periodic waves appear in the interaction region due to the resonant-type collision. The wave structures are very different from the wave structures of the standard collision between the two solitons displayed in Fig. 1, in which the periodic waves do not appear. Such resonant collision has been reported for the nonlocal M-NLS equations in Ref. [54], which are also generated by a phase shift tending to zero.

**iii Relative separation distance.** The changes in phase make the relative separation distance between the two solitons, which is generated by taking the position of soliton 2 minus the position of soliton 1. After algebraic calculations, we obtain that the relative separation distance between the two solitons are the same in the $u_1$ component and in the $u_2$ component:

\[
\Delta x_{12} = \frac{2}{p_1 R} \Phi_1^{(1)} = \frac{2}{p_1 R} \Phi_2^{(1)},
\]

(44)

and the relative separation distance between the two solitons with zero boundary condition in the $N$ component is

\[
\Delta \hat{x}_{12} = \frac{2}{p_1 R} \Phi_N^{(1)}.
\]

(45)

The expressions of $\Phi_1^{(1)}, \Phi_2^{(1)}, \Phi_N^{(1)}$ are defined in Eqs. (41), (42), from which we can infer that the relative separation distances [i.e., $\Delta x_{12}, \Delta \hat{x}_{12}$] in these three components [i.e., $u_1, u_2, N$] with different boundary conditions are determined by the parameters $p_1 R, p_1 I, \delta_1, \delta_2$. 
3.2. COLLISIONS OF BRIGHT-DARK FOUR SOLITONS

The bright-dark four-soliton solutions of the nonlocal 2CNNLS-Boussinesq equation are derived from Theorem 1 with $M = 2$, and the functions $f, g$ and $h$ of the bright-dark four-soliton solutions are

$$
\begin{align*}
    f &= \begin{vmatrix}
        m_{1,1}^{(0)} & m_{1,2}^{(0)} & m_{1,3}^{(0)} & m_{1,4}^{(0)} \\
        m_{2,1}^{(0)} & m_{2,2}^{(0)} & m_{2,3}^{(0)} & m_{2,4}^{(0)} \\
        m_{3,1}^{(0)} & m_{3,2}^{(0)} & m_{3,3}^{(0)} & m_{3,4}^{(0)} \\
        m_{4,1}^{(0)} & m_{4,2}^{(0)} & m_{4,3}^{(0)} & m_{4,4}^{(0)}
    \end{vmatrix}, \\
    g &= \begin{vmatrix}
        m_{1,1}^{(1)} & m_{1,2}^{(1)} & m_{1,3}^{(1)} & m_{1,4}^{(1)} \\
        m_{2,1}^{(1)} & m_{2,2}^{(1)} & m_{2,3}^{(1)} & m_{2,4}^{(1)} \\
        m_{3,1}^{(1)} & m_{3,2}^{(1)} & m_{3,3}^{(1)} & m_{3,4}^{(1)} \\
        m_{4,1}^{(1)} & m_{4,2}^{(1)} & m_{4,3}^{(1)} & m_{4,4}^{(1)}
    \end{vmatrix}, \\
    h &= \begin{vmatrix}
        m_{1,1}^{(0)} & m_{1,2}^{(0)} & m_{1,3}^{(0)} & m_{1,4}^{(0)} \\
        m_{2,1}^{(0)} & m_{2,2}^{(0)} & m_{2,3}^{(0)} & m_{2,4}^{(0)} \\
        m_{3,1}^{(0)} & m_{3,2}^{(0)} & m_{3,3}^{(0)} & m_{3,4}^{(0)} \\
        m_{4,1}^{(0)} & m_{4,2}^{(0)} & m_{4,3}^{(0)} & m_{4,4}^{(0)}
    \end{vmatrix} e^{p_1 x + ip_2 t} e^{\tilde{\eta}_1} e^{-p_1 x + ip_2 t} e^{-\tilde{\eta}_2} 0
\end{align*}
$$

where $m_{s,j}^{(n)}$ ($s, j = 1, 2, 3, 4; n = 0, 1$) are defined in Eq. (25). There are four free soliton parameters $p_s, \tilde{\eta}_s$ ($s = 1, 2$) and two nonlinear coefficients $\delta_s$ comprised in this bright-dark four-soliton solutions, and the parameters $p_s, \tilde{\eta}_s$ mainly control the identities of the pair of solitons.
Fig. 5 – (Colour online) The bound state two-soliton pair in the 2NLS-Boussinesq equation with $\delta_1 = 1, \delta_2 = 1, p_1 = 1 + i, p_2 = 3 - i, \tilde{\eta}_1 = \ln \sqrt{2} - \frac{i\pi}{4}, \tilde{\eta}_2 = \ln \sqrt{2} + \frac{i\pi}{4}$. The lower panels are the density plots of the upper panels.

Fig. 6 – (Colour online) The bound state two-soliton pair resonant collision in the 2NLS-Boussinesq equation with $\delta_1 = 1, \delta_2 = 1, p_1 = \frac{1}{10} + i, p_2 = \frac{1}{20} - i, \tilde{\eta}_1 = \ln \sqrt{2} - \frac{i\pi}{4}, \tilde{\eta}_2 = \ln \sqrt{2} + \frac{i\pi}{4}$. The lower panels are the density plots of the upper panels.
According to the classification of the bright-dark two-solitons upon their intensities, the two-soliton in the dark component (i.e., the \( u_1 \) component) has five patterns: \( \tilde{N} \)-dark–(4 – \( \tilde{N} \))-antidark four-solitons for \( \tilde{N} = 0, 1, 2, 3, 4 \); while the two-solitons in the bright components (i.e., the \( u_2 \) and \( N \) components) only has one pattern: bright four-solitons.

We point out that the two-soliton solutions only exhibit head on collision and do not admit bound state two-solitons. However, the four-solitons admit a pair of bound state two-solitons when \( p_1 I = \pm p_2 I \). A bound state two-solitons pair is demonstrated in Fig. 5 with parameters \( \delta_1 = 1, \delta_2 = 1, p_1 = 1 + i, p_2 = 3 - i, \tilde{\eta}_1 = \ln \sqrt{2} - \frac{i\pi}{4}, \tilde{\eta}_2 = \ln \sqrt{2} + \frac{i\pi}{4} \). It is inferred that the two-solitons pair features periodic waves in the \((x, t)\) plane. The dark component (i.e., the \( u_1 \) component) takes wave features as two breathers and two solitons in the \((x, t)\) plane, which is different from the usual four-solitons solutions. Similar to the two solitons which can exhibit resonant-type collision, the bound state two-soliton pair can also exhibit the resonant-type collision. In the region where the four solitons intersect with each other, the resonant-type collision generates irregular periodic wave. The resonant-type collision of the bound state two-solitons pair is shown in Fig. 6. An unique phenomenon is that, in the zero boundary condition component \( N \), four peaks appear in the intersection region, which are much higher than the intensity of the solitons far from the interaction region. The intensity profiles of the bound state two-solitons pair along time \( t = 2 \) are demonstrated in Fig. 7 with the same parameters as in Fig. 6, which reveal that the irregular periodic waves exist in the interaction region due to the resonant-type collision.

4. CONCLUSION AND DISCUSSION

In this paper, the 2CNNLS-Boussinesq equation (3) has been proposed and investigated due to the physical application of multicomponent nonlocal NLS/NLS-
type equations in nonlinear optics. The 2CNNLS-Boussinesq (3) comprises of three types of nonlinear combination, namely, focusing-focusing nonlinearities, defocusing-defocusing nonlinearities, and mixed focusing-defocusing nonlinearities, arising in various physical settings. The bright-dark $2M$-solitons, namely, one component (i.e., $u_1$ in Eq. (3)) featuring solitons with nonzero boundary condition and the other two components (i.e., $u_2, N$ in Eq. (3)) displaying solitons with zero boundary condition, have been systematically derived by employing the KP hierarchy reduction method. Based on the asymptotic analysis of the two-solitons, we have found that the dark component (i.e., the $u_1$ component) has three different types of non-degenerated two-solitons solutions and two different types of degenerated two-solitons, while the two bright components (i.e., the $u_2$ and $N$ components) only have one pattern of non-degenerated two-solitons solutions. The resonant-type collision between two-solitons has also been studied, which can generate periodic waves in the region where the two solitons interact with each other. For the bright-dark four-solitons, we have mainly considered the particular four-solitons namely, the bound state two-soliton pair. The bound state two-soliton pair exhibiting resonant-type collision has also been discussed in detail, which could give rise to a striking nonlinear phenomenon: the occurring of irregular periodic waves in the interaction region.

There are three possible extensions of the present work:

- The collision between these bright-dark $2M$-solitons is elastic, due to only one short-wave component (i.e., the $u_2$ component) featuring bright solitons. To investigate the energy-sharing collision of the $2M$-solitons in the 2CNNLS-Boussinesq equation, we have to construct the bright-bright $2M$-solitons, namely, the three components $u_1, u_2$, and $N$ all featuring bright solitons subject to zero boundary condition.

- The 2CNNLS-Boussinesq equation (3) only comprises of two component nonlocal NLS equation, thus another natural extension is to investigate solitons to the arbitrary $\hat{N} \geq 2$ components nonlocal NLS equation coupled to the Boussinesq equation, namely,

$$iu_{j,t} + u_{j,xx} + Nu_j = 0, \ j = 1, 2, \cdots \hat{N},$$

$$N_{xxxx} - 3(N^2)_{xx} - 3N_{tt} + N_{xx} + \left( \sum_{s=1}^{\hat{N}} \delta_{x}u_{x}u_{x}^{\ast}(-x,t) \right)_{xx} = 0. \quad (47)$$

- In Ref. [46], Yang proposed several integrable nonlocal NLS equations through constraining the Manakov system from physically significant points of views. Since the 2CNNLS-Boussinesq equation also contains two components nonlocal NLS equation, an extension of the 2CNNLS-Boussinesq equation (3) is
to investigate the soliton collisions in the following nonlocal NLS-Boussinesq equation of reverse-time/space types:

\[
\begin{align*}
    iu_t + u_{xx} + Nu &= 0, \\
    N_{xxxx} - 3(N^2)_{xx} - 3N_{tt} + N_{xx} + 2u \left[ \delta_1 |u|^2 + \delta_2 u^4 (\epsilon_1 x, \epsilon_2 t) \right]_{xx} &= 0,
\end{align*}
\]

(48)

where \(|\epsilon_1| = |\epsilon_2| = 1\), and \(\epsilon_1 \epsilon_2 = -1\) or \(\epsilon_1 = -1, \epsilon_2 = -1\).

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