Romanian Reports in Physics, Vol. 61, No. 4, P. 627–640, 2009

GENERAL PHYSICS

ENTANGLEMENT DYNAMICS IN OPEN SYSTEMS

AURELIAN ISAR

National Institute of Physics and Nuclear Engineering,
P.O. Box MG-6, RO-077125 Bucharest-Magurele, Romania
E-mail: isar@theory.nipne.ro

(Received August 28, 2009)

Abstract. In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we give a description of the continuous-variable entanglement for a system consisting of two uncoupled harmonic oscillators interacting with a general common environment. Using Peres-Simon necessary and sufficient criterion for separability of two-mode Gaussian states, we describe the generation and evolution of entanglement in terms of the covariance matrix for a Gaussian input state. For some values of diffusion and dissipation coefficients describing the environment, the state keeps for all times its initial type: separable or entangled. In other cases, entanglement generation, entanglement sudden death or a periodic collapse and revival of entanglement take place. We analyze also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state.

Key words: quantum entanglement, Gaussian states, open systems.

1. INTRODUCTION

In recent years there is an increasing interest in using continuous variable (CV) systems in applications of quantum information processing, communication and computation [1], like experimental observation of CV quantum teleportation [2] using a two-mode squeezed state, based on a CV theoretical description [3], the demonstration of quantum key distribution [4] for continuous optical fields, and the successful definition of the notion of universal quantum computation over CV [1]. The realization of quantum information processing tasks depends on the generation and manipulation of nonclassical states of CV systems. A full characterization of the nonclassical properties of entangled states of CV systems exists, at present, only for the class of Gaussian states. In this special case there exist necessary and sufficient criteria of entanglement [5, 6] and quantitative entanglement measures [7, 8]. In quantum information theory of CV systems, Gaussian states play a key role since they can be easily created and controlled experimentally.

Two-mode Gaussian states play an important role in quantum information processing tasks in CV systems, like quantum teleportation [1, 3], quantum cryptography [4] and quantum entanglement swapping [9]. Two-mode Gaussian entanglement has been generated in various physical situations like optical parametric amplifiers, nonlinear parametric down conversion, Kerr nonlinearity in an optical fiber or cavities, two-mode cavity quantum electrodynamics.

Quantum entanglement represents a key resource in quantum information processing. Implementation of quantum communication and computation encounters the difficulty that any realistic quantum system cannot be isolated and it always has to interact with its environment. Quantum coherence and entanglement of quantum systems are inevitably influenced during their interaction with the external environment. As a result of the irreversible and uncontrollable phenomenon of quantum decoherence, the purity and entanglement of quantum states are in most cases degraded. However, it was recently shown that entanglement can be created or enhanced during the interaction with the external environment, like, for example, in the case of a system of two non-interacting qubits coupled to a common environment [10]. At the same time there exist some special entangled states that are not altered by the interaction with the environment, called decoherence-free states that could be efficient in quantum information processing. Practically, compared with the discrete variable entangled states, the CV entangled states may be more efficient because they are less affected by decoherence.

Due to the unavoidable interaction with the environment, any pure quantum state evolves into a mixed state and to describe realistically CV quantum information processes it is necessary to take decoherence and dissipation into consideration. Decoherence and dynamics of quantum entanglement in CV open systems have bee intensively studied in the last years [11–31]. The Markovian time evolution of quantum correlations of entangled two-mode CV states has been examined in single-reservoir [10, 16] and two-reservoir models [6, 15, 17], representing noisy correlated or uncorrelated Markovian quantum channels.

When two systems are immersed in an environment, then, besides and at the same time with the quantum decoherence phenomenon, the environment can also generate a quantum entanglement of the two systems [19, 32]. In certain circumstances, the environment enhances the entanglement and in others it suppresses the entanglement and the state describing the two systems becomes separable. The structure of the environment may be such that not only the two systems become entangled, but also such that the entanglement is maintained for a definite time or a certain amount of entanglement survives in the asymptotic long-time regime. The effects of environment may include collapses and revivals of entanglement [33].

In the case of two modes of an electromagnetic field embedded in a thermal environment, in Ref. [16] it was derived a condition which states that if the state of the two modes is initially sufficiently squeezed, it will always remain entangled independent of the strength of the interaction with the environment. Studying the dynamics of two-mode squeezed states in an extended quantum Brownian motion model, Hörhammer et al. [34] showed that below a critical bath temperature, two-mode entanglement is preserved even in the steady state. Paz and Roncaglia [35] also analyzed the entanglement properties of two oscillators in a common environment by using the exact master equation for quantum Brownian motion and showed that the entanglement can undergo three phases: sudden death, sudden death and revival, and no sudden death.

In this paper we review the results obtained, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups, on the dynamics of the CV entanglement of two modes (two identical harmonic oscillators) coupled to a common environment characterized by general diffusion and dissipation coefficients [36–38]. We are interested in discussing the correlation effect of the environment, therefore we assume that the two systems are uncoupled, i.e. they do not interact directly. The initial state of the subsystem is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state.

The underlying approach assumes weak coupling between the system and the environment and neglects short-time correlations between the system and environment. This approach has been widely and successfully used in the field of quantum optics, where the characteristic time scales of the environmental correlations is much shorter compared to the internal system dynamics.

We show that both modes interact indirectly via the coupling to the environment. Therefore, new quantum correlations may emerge between the two modes and this model provides an example of environment-induced quantum two-mode entanglement.

The paper is organized as follows. In Sec. 2 we write the Markovian master equation in the Heisenberg representation for two uncoupled harmonic oscillators interacting with a general environment and the evolution equation for the covariance matrix. For this equation we give its general solution, i.e. we derive the variances and covariances of coordinates and momenta corresponding to a generic two-mode Gaussian state. In particular we derive the asymptotic values of the elements of the covariance matrix. By using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states [5, 39], we investigate in Sec. 3 the dynamics of entanglement for the considered subsystem. In particular, with the help of the asymptotic covariance matrix, we determine the behaviour of the entanglement in the limit of long times. We show that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the parameters describing the environment, the entanglement is suppressed and the asymptotic state is separable. The existence of the quantum correlations between the two systems in the asymptotic long-time regime is the result of the competition between entanglement and decoherence. We analyze also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state. This entanglement monotone of logarithmic negativity is conveniently computable for general Gaussian states, and it provides a proper quantification of entanglement in particular for two-mode Gaussian states. A summary and conclusions are given in Sec. 4.

2. EQUATIONS OF MOTION FOR TWO HARMONIC OSCILLATORS

We study the dynamics of the subsystem composed of two identical noninteracting oscillators in weak interaction with a general environment. In the axiomatic formalism based on completely positive quantum dynamical semigroups, the irreversible time evolution of an open system is described by the following general quantum Markovian master equation for an operator A in the Heisenberg representation († denotes Hermitian conjugation) [40, 41]:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}=\frac{i}{\hbar}\left[H,\ A(t)\right]+\frac{1}{2\hbar}\sum\left(V_j^{\dagger}\left[A(t),V_j\right]+\left[V_j^{\dagger},A(t)\right]V_j\right). \qquad (1)$$

Here, H denotes the Hamiltonian of the open system and the operators $V_j, V_j^{\dagger}$, defined on the Hilbert space of H, represent the interaction of the open system with the environment.

We are interested in the set of Gaussian states, therefore we introduce such quantum dynamical semigroups that preserve this set during time evolution of the system and in this case our model represents a Gaussian noise channel. Consequently H is taken to be a polynomial of second degree in the coordinates x, y and momenta p_x, p_y of the two quantum oscillators and $V_j, V_j^{\dagger}$ are taken polynomials of first degree in these canonical observables. Then in the linear space spanned by the coordinates and momenta there exist only four linearly independent operators $V_{j=1,2,3,4}$ [42]:

$$V_j = a_{xj}p_x + a_{yj}p_y + b_{xj}x + b_{yj}y, \qquad (2)$$

where a_{xj}, a_{yj}, b_{xj}, b_{yj} are complex coefficients. The Hamiltonian H of the two uncoupled identical harmonic oscillators of mass m and frequency ω is given

$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+\frac{m\omega^2}{2}\left(x^2+y^2\right). \qquad (3)$$

The fact that the evolution is given by a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors $\mathbf{a}_x$, $\mathbf{b}_x$, $\mathbf{a}_y$, $\mathbf{b}_y$ whose entries are the components a_{xj}, b_{xj}, a_{yj}, b_{yj}, respectively:

$$\frac{1}{2}\hbar\begin{pmatrix} (\mathbf{a}_x\mathbf{a}_x) & (\mathbf{a}_x\mathbf{b}_x) & (\mathbf{a}_x\mathbf{a}_y) & (\mathbf{a}_x\mathbf{b}_y) \\ (\mathbf{b}_x\mathbf{a}_x) & (\mathbf{b}_x\mathbf{b}_x) & (\mathbf{b}_x\mathbf{a}_y) & (\mathbf{b}_x\mathbf{b}_y) \\ (\mathbf{a}_y\mathbf{a}_x) & (\mathbf{a}_y\mathbf{b}_x) & (\mathbf{a}_y\mathbf{a}_y) & (\mathbf{a}_y\mathbf{b}_y) \\ (\mathbf{b}_y\mathbf{a}_x) & (\mathbf{b}_y\mathbf{b}_x) & (\mathbf{b}_y\mathbf{a}_y) & (\mathbf{b}_y\mathbf{b}_y) \end{pmatrix}. \tag{4}$$

We take this matrix of the following form, where all coefficients D_{xx}, D_{xp_x}, ... and λ are real quantities (we put from now on $\hbar = 1$):

$$\begin{pmatrix} D_{xx} & -D_{xp_x} - i\lambda/2 & D_{xy} & -D_{xp_y} \\ -D_{xp_x} + i\lambda/2 & D_{p_xp_x} & -D_{yp_x} & D_{p_xp_y} \\ D_{xy} & -D_{yp_x} & D_{yy} & -D_{yp_y} - i\lambda/2 \\ -D_{xp_y} & D_{p_xp_y} & -D_{yp_y} + i\lambda/2 & D_{p_yp_y} \end{pmatrix}. \tag{5}$$

It follows that the principal minors of this matrix are positive or zero. From the Cauchy-Schwarz inequality the following relations hold for the coefficients defined in Eq. (5):

$$\begin{gathered} D_{xx}D_{p_xp_x} - D_{xp_x}^2 \ge \frac{\lambda^2}{4}, \quad D_{yy}D_{p_yp_y} - D_{yp_y}^2 \ge \frac{\lambda^2}{4}, \\ D_{xx}D_{yy} - D_{xy}^2 \ge 0, \quad D_{p_xp_x}D_{p_yp_y} - D_{p_xp_y} \ge 0, \\ D_{xx}D_{p_yp_y} - D_{xp_y}^2 \ge 0, \quad D_{yy}D_{p_xp_x} - D_{yp_x}^2 \ge 0. \end{gathered} \tag{6}$$

The matrix of the coefficients (5) can be conveniently written as (T denotes the transposed matrix)

$$\begin{pmatrix} C_1 & C_3 \\ C_3^{\mathrm{T}} & C_2 \end{pmatrix}, \tag{7}$$

in terms of 2 × 2 matrices $C_1 = C_1^\dagger$, $C_2 = C_2^\dagger$ and C_3. This decomposition has a direct physical interpretation: the elements containing the diagonal contributions C_1 and C_2 represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in C_3 represent environment generated couplings between the two oscillators, taken initially independent.

We introduce the following 4 × 4 bimodal covariance matrix:

$$\sigma(t)=\begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{p_xp_x}(t) & \sigma_{yp_x}(t) & \sigma_{p_xp_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_x}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{p_xp_y}(t) & \sigma_{yp_y}(t) & \sigma_{p_yp_y}(t) \end{pmatrix}, \tag{8}$$

with the correlations of operators A_1 and A_2, defined by using the density operator ρ of the initial state of the quantum system, as follows:

$$\sigma_{A_1A_2}(t)=\frac{1}{2}\mathrm{Tr}\left[\rho\left(A_1A_2+A_2A_1\right)(t)\right]-\mathrm{Tr}\left[\rho A_1(t)\right]\mathrm{Tr}\left[\rho A_2(t)\right]. \tag{9}$$

Using this definition, we can transform the problem of solving the master equation for the operators in Heisenberg representation into a problem of solving first-order in time, coupled linear differential equations for the covariance matrix elements. Namely, from Eq. (1) we obtain the following system of equations for the quantum correlations of the canonical observables [42]:

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}=Y\sigma(t)+\sigma(t)Y^{\mathrm{T}}+2D, \tag{10}$$

where

$$Y=\begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1/m \\ 0 & 0 & -m\omega^2 & -\lambda \end{pmatrix}, \tag{11}$$

$$D=\begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_y} & D_{p_xp_x} & D_{yp_x} & D_{p_xp_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_xp_y} & D_{yp_y} & D_{p_yp_y} \end{pmatrix}. \tag{12}$$

The time-dependent solution of Eq. (10) is given by [42]

$$\sigma(t)=M(t)\left[\sigma(0)-\sigma(\infty)\right]M^{\mathrm{T}}(t)+\sigma(\infty), \tag{13}$$

where the matrix $M(t) = \exp(Yt)$ has to fulfill the condition $\lim_{t\to\infty} M(t) = 0.$ In order that this limit exists, Y must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation

$$Y\sigma(\infty) + \sigma(\infty)Y^{\mathrm{T}} = -2D. \tag{14}$$

3. DYNAMICS OF TWO-MODE CONTINUOUS VARIABLE ENTANGLEMENT

In the following we review the quantum separability criteria for a special class of states of CV systems – the two-mode Gaussian states. A Gaussian two-mode state has a Gaussian Wigner function in phase space and it is completely characterized by its first and second moments of canonical variables. A well-known sufficient condition for inseparability is the so-called Peres-Horodecki criterion [39, 43] which is based on the observation that the noncompletely positive nature of the partial transposition operation of the density matrix for a bipartite system (this means transposition with respect to degrees of freedom of one subsystem only) may turn an inseparable state into a nonphysical state. The signature of this nonphysicality, and thus of quantum entanglement, is the appearance of a negative eigenvalue in the eigenspectrum of the partially transposed density matrix of a bipartite system. The characterization of the separability of CV states using second-order moments of quadrature operators was given in Refs. [5, 6]. For Gaussian states, whose statistical properties are fully characterized by just second-order moments, this criterion was proven to be necessary and sufficient: A Gaussian CV state is separable if and only if the partial transpose of its density matrix is nonnegative [positive partial transpose (PPT) criterion].

The two-mode Gaussian state is entirely specified by its covariance matrix (8), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma(t) = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{15}$$

where A, B and C are 2×2 Hermitian matrices. Their entries are correlations of the canonical operators x, y, p_x and p_y; A and B denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix C contains the cross-correlations between modes.

The 4×4 covariance matrix (15) (where all first moments have been set to zero by means of local unitary operations which do not affect the entanglement) contains four local symplectic invariants in form of the determinants of the block matrices A, B, C and covariance matrix σ. Based on the above invariants Simon

[5] derived a PPT criterion for bipartite Gaussian CV states: the necessary and sufficient criterion for separability is $S(t) \geq 0$, where

$$S(t) \equiv \det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \mathrm{Tr}\left[AJCJBJC^{\mathrm{T}}J\right] - \frac{1}{4}\left(\det A + \det B\right) \quad (16)$$

and J is the 2 × 2 symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (17)$$

This is also a necessary separability criterion for non-Gaussian states. For a Gaussian two-mode state the partial transpose implies a mirror reflection in one of the two momenta operators and this leads to a change of the signs in those elements of the covariance matrix, which connect the momentum of one mode to the coordinate of the other mode.

The elements of the covariance matrix depend on Y and D and can be calculated from Eqs. (13), (14). Solving for the time evolution of the covariance matrix elements, we can obtain the entanglement dynamics through the computation of the Simon criterion or by calculating logarithmic negativity, as will be shown in the following. Since the two oscillators are identical, it is natural to consider environments for which the two diagonal submatrices in Eq. (7) are equal, $C_1 = C_2$, and the matrix C_3 is symmetric, so that in the following we take $D_{xx} = D_{yy}$, $D_{xp_x} = D_{yp_y}$, $D_{p_x p_x} = D_{p_y p_y}$, $D_{xp_y} = D_{yp_x}$. Then both unimodal covariance matrices are equal, $A = B$, and the entanglement matrix C is symmetric.

3.1. TIME EVOLUTION OF ENTANGLEMENT

It is interesting that the general theory of open quantum systems allows couplings via the environment between uncoupled oscillators. According to the definitions of the environment parameters, the diffusion coefficients can take non-zero values and therefore can simulate an interaction between the uncoupled oscillators. Consequently, the cross-correlations between modes can have non-zero values. In this case the Gaussian states with $\det C \geq 0$ are separable states, but for $\det C < 0$ it may be possible that the states are entangled.

In order to describe the dynamics of entanglement, we use the PPT criterion [5, 39] according to which a state is entangled if and only if the operation of partial transposition does not preserve its positivity. Concretely, we have to analyze the time evolution of the Simon function $S(t)$ (16). We consider two cases, according to the type of the initial Gaussian state: separable or entangled. A special subclass of entangled two-mode Gaussian states is given by the so-called two-mode squeezed vacuum states. For a zero squeezing parameter, two-mode state corresponds to a separable (disentangled) state. Two-mode squeezed vacuum states are usually generated in quantum optics laboratories and have been used in most implementations of CV quantum information protocols.

In the following we consider such environment diffusion coefficients, for which

$$m^2\omega^2 D_{xx} = D_{p_x p_x}, D_{yp_y} = 0, m^2\omega^2 D_{xy} = D_{p_x p_y} \qquad (18)$$

This corresponds to the case when the asymptotic state is a Gibbs state [41].

1) To illustrate a possible generation of the entanglement, we analyzed in [36–38] the dependence of function $S(t)$ on time t and diffusion coefficient $D_{xp_y} \equiv d$ for a separable initial Gaussian state. We obtained that, according to Peres-Simon criterion, for relatively small values of the coefficient d, the initial separable state remains separable for all times. For larger values of d, at some finite moment of time, when $S(t)$ becomes negative, the state becomes entangled.

In the case of a generated entanglement we notice several situations: a) the entanglement is created only for a short time, then it disappears and the state becomes again separable; b) there exist repeated collapse and revival of entanglement; c) entanglement may persist forever, including the asymptotic final state. These situations depend on the coefficients characterizing the environment.

The entanglement of the two modes can be generated from an initial separable state during the interaction with the environment only for certain values of diffusion and dissipation coefficients.

2) The evolution of an entangled initial state is also described in Refs. [36–38], where we analyzed the dependence of function $S(t)$ on time t and diffusion coefficient d. We observed that for relatively small values of d, at some finite moment of time $S(t)$ takes non-negative values and therefore the state becomes separable. This is the so-called phenomenon of entanglement sudden death. This phenomenon is in contrast to the loss of quantum coherence, which is usually gradual [29, 44]. Depending on the values of the coefficient d, it is also possible to have a repeated collapse and revival of the entanglement. For relatively large values of the coefficients D_{xx} and d, the initial entangled state remains entangled for all times.

The dynamics of entanglement of the two oscillators depends strongly on the initial states and the coefficients describing the interaction of the system with the environment.

3.2. ASYMPTOTIC ENTANGLEMENT

On general grounds, one expects that the effects of decoherence, counteracting entanglement production, is dominant in the long-time regime, so that no quantum correlations (entanglement) is expected to be left at infinity.

Nevertheless, we have seen previously that there are situations in which the environment allows the presence of entangled asymptotic equilibrium states. From Eq. (14) we obtain the following elements of the asymptotic entanglement matrix $C(\infty)$:

$$\sigma_{xy}(\infty)=\frac{m^2\left(2\lambda^2+\omega^2\right)D_{xy}+2m\lambda D_{xp_y}+D_{p_xp_y}}{2m^2\lambda\left(\lambda^2+\omega^2\right)}, \tag{19}$$

$$\sigma_{xp_y}(\infty)=\sigma_{yp_x}(\infty)\frac{-m^2\omega^2D_{xy}+2m\lambda D_{xp_y}+D_{p_xp_y}}{2m\left(\lambda^2+\omega^2\right)}, \tag{20}$$

$$\sigma_{p_xp_y}(\infty)=\frac{m^2\omega^4D_{xy}-2m\omega^2\lambda D_{xp_y}+\left(2\lambda^2+\omega^2\right)D_{p_xp_y}}{2\lambda\left(\lambda^2+\omega^2\right)}. \tag{21}$$

The corresponding elements of matrices $A(\infty)$ and $B(\infty)$ are obtained by setting $x = y$ in the previous expressions. We calculate the determinant of the entanglement matrix and obtain:

$$\det C(\infty)=\frac{1}{4\lambda^2\left(\lambda^2+\omega^2\right)}\left[\left(m\omega^2D_{xy}+\frac{1}{m}D_{p_xp_y}\right)^2+4\lambda^2\left(D_{xy}D_{p_xp_y}-D_{xp_y}^2\right)\right]. \tag{22}$$

With the chosen coefficients (18), the Simon expression (16) takes the following form in the limit of large times:

$$S(\infty)=\left(\frac{m^2\omega^2\left(D_{xx}^2-D_{xy}^2\right)}{\lambda^2}+\frac{D_{xp_y}^2}{\lambda^2+\omega^2}-\frac{1}{4}\right)^2-4\frac{m^2\omega^2D_{xy}^2D_{xp_y}^2}{\lambda^2\left(\lambda^2+\omega^2\right)}. \tag{23}$$

For environments characterized by such coefficients that the expression $S(\infty)$ (23) is strictly negative, the asymptotic final state is entangled. In particular, for $D_{xy} = 0$ we obtain that $S(\infty) < 0$, i.e. the asymptotic final state is entangled, for the following range of values of the coefficient D_{xp_y} characterizing the environment [44, 45]:

$$\frac{m\omega D_{xx}}{\lambda}-\frac{1}{2}<\frac{D_{xp_y}}{\sqrt{\lambda^2+\omega^2}}<\frac{m\omega D_{xx}}{\lambda}+\frac{1}{2}, \tag{24}$$

where the diffusion coefficient D_{xx} satisfies the condition $m\omega D_{xx}/\lambda\geq 1/2$, equivalent with the unimodal uncertainty relation. We remind that, according to inequalities (6), the coefficients have to fulfill also the constraint $D_{xx}\geq D_{xp_y}$. If the coefficients do not fulfil the inequalities (24), then $S(\infty)\geq 0$ and the asymptotic state of the considered system is separable.

3.3. LOGARITHMIC NEGATIVITY

In order to quantify the degrees of entanglement of the infinite-dimensional bipartite system states of the two oscillators it is suitable to use the logarithmic negativity. The logarithmic negativity of a bipartite system consisting of two subsystems A and B is [7] $E_N(\rho) = \log_2 \left\|\rho^{T_B}\right\|_1$, where ρ^{T_B} means the partial transpose of a mixed state density matrix operator ρ with respect to subsystem B. The operation $\|\cdot\|_1$ denotes the trace norm, which for any Hermitian operator O is defined as $\|O\|_1 \equiv \mathrm{Tr}|O| \equiv \mathrm{Tr}\sqrt{O^\dagger O}$ and it is calculated as the sum of absolute values of the eigenvalues of O.

Logarithmic negativity quantifies the degree of violation of PPT criterion for separability, i.e. how much the partial transposition of ρ fails to be positive and it is based on negative eigenvalues of the partial transpose of the subsystem density matrix. For a Gaussian density operator, the negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix and it is given by

$$E_N = \max\left\{0, -\log_2 2\tilde{\nu}_-\right\}, \tag{25}$$

where $\tilde{\nu}$ is the smallest of the two symplectic eigenvalues of the partial transpose $\tilde{\sigma}$ of the 2-mode covariance matrix σ:

$$2\tilde{\nu}_\mp^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - 4\det\sigma}. \tag{26}$$

Here $\tilde{\Delta}$ is the symplectic invariant (seralian), given by

$$\tilde{\Delta} = \det A + \det B + 2\det\tilde{C} = \det A + \det B - 2\det C = \Delta - 4\det C. \tag{27}$$

Logarithmic negativity E_N is a decreasing function of $\tilde{\nu}_-$ only and quantifies the amount by which inequality $\tilde{\nu}_- \geq 1/2$ is violated (this inequality is equivalent to the Simon criterion of separability). As $\tilde{\nu}_-$ goes to 0, logarithmic negativity diverges, while it becomes zero for $\tilde{\nu}_- \geq 1/2$, and the state is separable. Despite not being convex, the logarithmic negativity is a full entanglement monotone under local operations and classical communication (LOCC). Therefore the smallest partially transposed symplectic eigenvalue $\tilde{\nu}_-$ alone completely determines qualitatively and quantitatively the quantum entanglement of a two-mode Gaussian state [46]. That is, the smaller the value of $\tilde{\nu}_-$, the more entangled the corresponding two-mode Gaussian state.

In our model, the logarithmic negativity is calculated as

$$E_N(t) = -\frac{1}{2}\log_2\left[4f\left(\sigma(t)\right)\right],$$

where

$$f\left(\sigma(t)\right) = \frac{1}{2}\left(\det A + \det B\right) - \det C - \\ - \left(\left[\frac{1}{2}\left(\det A + \det B\right) - \det C\right]^2 - \det\sigma(t)\right)^{1/2}. \quad (28)$$

It determines the strength of entanglement for $E_N(t) \le 0$ and if $E_N(t) \ge 0$, then the state is separable. In Refs. [36–38, 44, 45, 47] we described the dependence of the logarithmic negativity $E_N(t)$ on time and diffusion coefficient D_{xp_y} for the two types of the initial Gaussian state, separable or entangled, previously considered when we analyzed the time evolution of the Simon function $S(t)$. As expected, the logarithmic negativity has a behaviour similar to that one of the Simon function in what concerns the characteristics of the state of being separable or entangled. Depending on the values of the environment coefficients, the initial state can preserve for all times its initial property – separable or entangled, and we can also notice the generation of entanglement or the collapse of entanglement (entanglement sudden death) at those finite moments of time when the logarithmic negativity $E_N(t)$, strictly positive initially, reaches zero value. One can also observe a repeated collapse and revival of the entanglement. In the case of an entangled initial state, for certain values of environment coefficients the logarithmic negativity is a fluctuating function which decreases asymptotically in time. When λ increases, then the oscillations of the logarithmic negativity are attenuated, i.e. for smaller λ we have more oscillations.

The asymptotic logarithmic negativity has the form

$$E_N = -\log_2\left[2\left|\frac{m\omega D_{xx}}{\lambda} - \frac{D_{xp_y}}{\sqrt{\lambda^2+\omega^2}}\right|\right]. \quad (29)$$

It depends only on the diffusion and dissipation coefficients characterizing the environment and does not depend on the initial Gaussian state.

4. SUMMARY AND CONCLUSIONS

In the framework of the theory of open quantum systems based on completely positive quantum dynamical semigroups, we investigated the Markovian dynamics of the quantum entanglement for a subsystem composed of two noninteracting modes embedded in a general common environment. We have presented and discussed the influence of the environment on the entanglement dynamics for different initial states. By using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states, we have described the generation and evolution of entanglement in terms of the covariance matrix for Gaussian input states. For some values of diffusion and dissipation coefficients describing the environment, the state keeps for all times its initial type: separable or entangled. In other cases, entanglement generation or entanglement suppression (entanglement sudden death) take place or even one can notice a repeated collapse and revival of entanglement. The dynamics of the quantum entanglement is sensitive to the initial states and the coefficients characterizing the environment.

We have also shown that, independent of the type of the initial state, for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the coefficients describing the environment, the asymptotic state is separable.

We described also the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state. We determined the range of diffusion and dissipation coefficients for entanglement to exist at long times for the two noninteracting subsystems coupled to the common environment.

In conclusion, the common environment can provide an indirect coupling between the two subsystems and therefore a mechanism to correlate them. Thus the coupling to the environment induces not only decoherence leading to separability of states, but also can generate quantum correlations.

The existence of quantum correlations between the two uncoupled harmonic oscillators interacting with a common environment is the result of the competition between the environment-induced entanglement and environmentinduced quantum decoherence. From the formal point of view, the generation of entanglement or its suppression (entanglement sudden death) correspond to the finite time vanishing of the Simon separability function or, respectively, of the logarithmic negativity. Presently there is a large debate relative to the physical interpretation existing behind these fascinating phenomena. Due to the increased interest manifested towards the CV approach to quantum information theory, the presented results, in particular the possibility of maintaining a bipartite entanglement in a diffusive-dissipative environment for asymptotic long times, might be useful in controlling the entanglement in open systems and also for applications in quantum information processing and communication.

Acknowledgements. The author acknowledges the financial support received within the Project IDEI 497/2009 and Project PN 09 37 01 02/2009.

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