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THE TRANSMISSION COEFFICIENT OF GRAVITO-ACOUSTIC WAVES IN NON-MAGNETIZED STRATIFIED PLASMA

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Abstract. In this parer I study transmission properties of gravito-acoustic waves at a horizontal interface separating two isothermal regions of a gravitationally stratified non-magnetized plasma. Possible applications to the boundary between the solar interior and the corona are discussed.

Key words: hydrodynamics, gravito-acoustic waves, Sun.

1. INTRODUCTION

The wave propagation in a compressible medium stratified in gravitational field, such as the atmosphere of the Sun or similar stars [1] is considered. In such a medium, without magnetic field, waves can be driven by both compressional and buoyancy restoring forces and each of them can induce harmonic oscillations of a fluid element slightly displaced from its equilibrium position. As a result the behaviour of waves is more complicated than in a homogeneous medium: $1)$ Their propagation characteristics are anisotropic because the gravity force imposes a preferred direction in the fluid. $2)$ They are dispersive (*i.e.* the propagation speed varies as k and/or ω are varied). $3)$ Stratification of the atmosphere imposes a cutoff frequency below which gravity-modified acoustic waves cannot propagate, and thus restricts the region of $(k_p - \omega)$ space in which such waves can exist. $4)$ Buoyancy forces give rise to a new class of propagating waves, the low-frequency pressure-modified gravity waves.

In a non-isothermal atmosphere propagating waves experience both reflection and transmission. In this paper are analysed waves that propagate in the region (1)-photosphere and can propagate or be evanescent in the region (2)-corona. I derived the transmission coefficient of gravito-acoustic waves because it could be the way for better understanding mechanical energy transport in the context of heating a stellar atmosphere [2,3]. Adiabatic fluctuations are considered because the effect of viscosity and thermal conduction on gravito-acoustic waves are negligible in astrophysical media.

2. BASIC EQUATIONS

The standard set of hydrodynamic equations describing the dynamics of adiabatic processes in a fully ionized hydrogen plasma in presence of gravity with constant acceleration is:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \\ \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \rho \vec{g}, \\ \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = \gamma \frac{p}{\rho} \left(\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho \right). \end{aligned} \tag{1}$$

The unperturbed gas is initially in hydrostatic equilibrium and assumed to be stepwise isothermal $T_0 = const$, *i.e.* with constant speed of sound v_s in each of the two regions separated by the boundary $z = 0$. The basic state is thus described by:

$$\frac{d}{dz} \ln \rho_0 = -\frac{\gamma g}{v_s^2}, \tag{2}$$

with $v_s^2 = \gamma R T_0$, $\gamma = c_p/c_v$ being the ratio of specific heats: $R = R_0/\bar{M}$ where $R_0 = 8.3145\ JK^{-1}mol^{-1}$ is the universal gas constant and $\bar{M} = 0.5\ kg\ mol^{-1}$ is the mean particle molar mass of the considered e-p plasma.
Eq. (2) now yields the solution for density profile:

$$\rho_0(z) = \rho_{00} e^{-z/H}, \tag{3}$$

where

$$H = \frac{v_s^2}{\gamma g}$$

is the constant scale height. The described basic state quantities Ψ_0 are subject to small adiabatic perturbations $\Psi_1(x,y,z,t)$ that are harmonic in time t and in coordinates x and y, whose amplitudes $\hat{\Psi}$ satisfy the condition $|\hat{\Psi}| \ll |\Psi_0|$ and depend on the vertical coordinate z:

$$\Psi_1(x,y,z,t) = \hat{\Psi}(k_x,k_y,\omega;z) e^{i(k_x x + k_y y - \omega t)}. \tag{4}$$

The standard set of ideal hydrodynamic equations for such perturbations can be reduced to a sistem of two coupled ordinary differential equations:

$$\begin{aligned} \frac{d\hat{\xi_{1z}}}{dz} = C_1 \hat{\xi_{1z}} - C_2 \hat{p_1} \\ \frac{d\hat{p_1}}{dz} - g\frac{d\rho_0}{dz}\hat{\xi_{1z}} = C_3 \hat{\xi_{1z}} - C_1 \hat{p_1}, \end{aligned} \tag{5}$$

where $\hat{\xi_{1z}} = i\hat{v_{1z}}/\omega$ is the z-component (*i.e.* the vertical component) of the Lagrangian displacement, while $\hat{p_1}$ is pressure perturbation. The coefficients in equations (5) are:

$$C_1 = \frac{g}{v_s^2}$$

$$C_2 = \frac{\omega^2 - k_p^2 v_s^2}{\rho_0(z) v_s^2 \omega^2}, \tag{6}$$

$$C_3 = \rho_0(z)\left(\omega^2 + \frac{g^2}{v_s^2}\right).$$

The density distribution $\rho_0(z)$ is given by (3) and $k_p^2 = k_x^2 + k_y^2$ designate squares of the horizontal wave-number. The equations Eqs.(5)-(6) allow the following solutions for the vertical displacement $\hat{\xi_{1z}}$ and the pressure perturbation $\hat{p_1}$:

$$\hat{\xi_{1z}}(z) = \xi_{1z}(0) e^{\frac{z}{2H}} e^{ik_z z}, \tag{7}$$

$$\hat{p_1}(z) = p_1(0) e^{\frac{-z}{2H}} e^{ik_z z}. \tag{8}$$

Eqs. (5) can further be transformed to a system of equations with constant coefficients if expressions: $\hat{\xi_{1z}} e^{-\frac{z}{2H}}$ and $\hat{p_1} e^{\frac{z}{2H}}$ are introduced as new unknowns. This finally yield the dispersion equation:

$$k_p^2 = \frac{\omega^2(\omega^2 - \omega_{co}^2 - v_s^2 k_z^2)}{v_s^2(\omega^2 - \omega_{BV}^2)}, \tag{9}$$

where

$$\omega_{co}^2 = \frac{\gamma^2 g^2}{4 v_s^2} = \frac{v_s^2}{4H^2}$$

is the acoustic wave cut-off frequency squared and

$$\omega_{BV}^2 = \frac{(\gamma - 1) g^2}{v_s^2} = \frac{(\gamma - 1) v_s^2}{\gamma^2 H^2}$$

is the squared Brunt-V$\ddot{a}$is$\ddot{a}$l$\ddot{a}\ddot{a}$ frequency. Equation (9) is fourth-order in ω and it corresponds to the gravito-acoustic wave equation known from the literature [1], [4]. Physical quantities in dispersion equation can be made dimensionless by appropriate scalings. Introducing dimensionless parameters: $K_p = k_p H$, $K_z = k_z H$ and $\Omega = \omega H / v_s$ where K_p and K_z are dimensionless horizontal and vertical wavenumbers scaled to $1/H$, while Ω is dimensionless frequency scaled to v_s/H, dispersion equation (9) gets the dimensionless form:

$$K_p^2 = \frac{\Omega^2(\Omega^2 - K_z^2 - \Omega_{co}^2)}{\Omega^2 - \Omega_{BV}^2}, \tag{10}$$

where $\Omega_{co} = 1/2$ and $\Omega_{BV} = \sqrt{\gamma - 1/\gamma^2}$. Fig. 1 for $\Omega = f(K_p)$, where K_z is a dimensionless parameter, illustrate solutions of the gravito-acoustic wave dispersion equation.

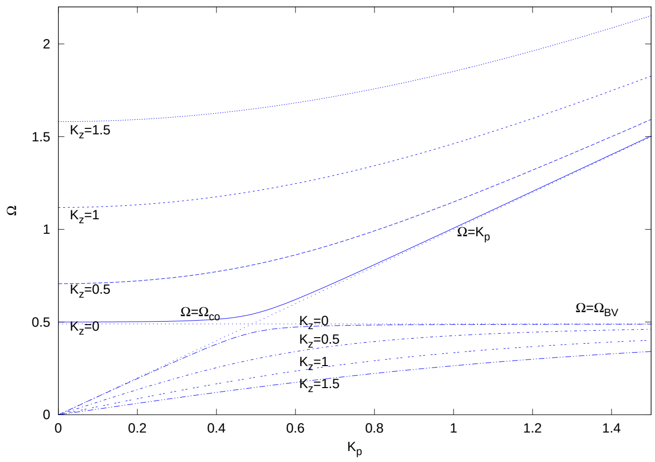

Fig. 1 – Dispersion equation for gravito-acoustic waves in an isothermal atmosphere.

3. BOUNDARY CONDITIONS

A plane, monochromatic wave propagating from one homogeneous medium (1) with constant speed of sound v_{s1} into a second homogeneous medium (2) with constant speed of sound v_{s2}, separated by a horizontal boundary plane $z = 0$ is considered. Plasma densities are ρ_{01} and ρ_{02} adjacent to the lower and upper side of the boundary *i.e.* at $z = 0 \pm \delta$. Thus the unperturbed density profile $\rho_0(z)$ is expressed by the following equation:

$$\rho_0(z) = \rho_{01} e^{-z/H_1}; z < 0 \tag{11}$$

and

$$\rho_0(z) = \rho_{02} e^{-z/H_2}; z > 0. \tag{12}$$

The stratification is defined by this equations. At the discontinuity, which has to be considered like the interface between two different non-mixing fluids through which there is no mass flux, the wave motion must satisfy the boundary conditions. Boundary condition that has to be applied is the equality of the unperturbed pressures at $z = 0$, yields:

$$\rho_{01} v_{s1}^2 = \rho_{02} v_{s2}^2. \tag{13}$$

I introduced parameter $s = \rho_{02}/\rho_{01} = v_{s1}^2/v_{s2}^2 = T_1/T_2$, which is constant because mediums (1)-photosphere and (2)-corona are isothermal with temperatures $T_1 = 6 \cdot 10^3 K$ and $T_2 = 10^6 K$ respectively. This means that $s = 0.006$ and it will be used later.
Boundary conditions for perturbations are:

$$\hat{\xi_{1z}}(\delta) = \hat{\xi_{1z}}(-\delta). \tag{14}$$

and

$$\hat{p_1}(\delta) - g\rho_{02}\hat{\xi_{1z}}(\delta) = \hat{p_1}(-\delta) - g\rho_{01}\hat{\xi_{1z}}(-\delta). \tag{15}$$

Here, $\hat{p_1} = \frac{C_{1n}}{C_{2n}}\hat{\xi_{1z}} - \frac{1}{C_{2n}}\frac{d\hat{\xi_{1z}}}{dz}$, as follows from the first Eq. in (5) for the regions $n = 1, 2$. Physical significance of the boundary conditions (14) and (15) are continuity of both the vertical Lagrangean displacement $\hat{\xi_{1z}}$ and the entire pressure perturbation $\hat{p_1} - g\rho_0\hat{\xi_{1z}}$ at the boundary $z = 0$. Here, the entire pressure perturbation results from two sources: the thermal (kinetic) pressure and an additional pressure arising from the gravity force.

It is known that a harmonic wave, which propagates through regions (1) and (2), does not change its dimensionless frequency $\Omega = \omega H_1/v_{s1}$ and horizontal dimensionless wavevector component $K_p = k_p H_1$, that is parallel to the boundary $z = 0$ [5]. However, the normal dimensionless wavevector component K_z has a discontinuity at the boundary $z = 0$, where it changes from K_{z1} to K_{z2} according to the dispersion relation (10). Here, $K_{z1} = k_{z1}H_1$ and $K_{z2} = k_{z2}H_1$. Let an incident wave with a unit amplitude in region (1) be partially reflected at the boundary $z = 0$ with amplitude A_r and partially transmitted into the region (2) with amplitude A_t. In this case, the solutions for the displacement $\hat{\xi_{1z}}$, according to equation (7), can be written as:

$$\hat{\xi_{1z}} = e^{[iK_{z1}+\frac{1}{2}]\frac{z}{H_1}} + A_r e^{[-iK_{z1}+\frac{1}{2}]\frac{z}{H_1}}, z < 0 \tag{16}$$

and

$$\hat{\xi_{1z}} = A_t e^{[iK_{z2}+\frac{s}{2}]\frac{z}{H_1}}, z > 0. \tag{17}$$

Eqs. (16) and (17) show that in the lower region perturbations are superposition of the incident and reflected waves, while in the upper region there is only transmitted wave. The solution for $\hat{p_1}$, could be rewritten in the form:

$$\hat{p_{1n}} = g\Pi_{1n}e^{-z/H_n}\hat{\xi_{1z}} - \Pi_{2n}e^{-z/H_n}\frac{d\hat{\xi_{1z}}}{dz}, \tag{18}$$

where:

$$\Pi_{1n} = \frac{\rho_{0n}s^{(n-1)}V_h^2}{s^{(n-1)}V_h^2 - 1} \tag{19}$$

and

$$\Pi_{2n} = \frac{\rho_{0n}v_{sn}^2 s^{(n-1)}V_h^2}{s^{(n-1)}V_h^2 - 1}, \tag{20}$$

with dimensionless horizontal phase velocity $V_h = \Omega/K_p$, or, according to equation (8):

$$\hat{p_1} = G_{(1,1)} e^{[iK_{z1} - \frac{1}{2H_1}]z} + A_r G_{(1,2)} e^{-[iK_{z1} + \frac{1}{2H_1}]z}, z < 0 \tag{21}$$

$$\hat{p_1} = A_t G_{(2,1)} e^{[iK_{z2} - \frac{s}{2}]\frac{z}{H_1}}, z > 0, \tag{22}$$

Here,

$$\begin{aligned} G_{(1,1)} &= g\Pi_{11} - \left[iK_{z1} + \frac{1}{2}\right]\frac{\Pi_{21}}{H_1} = \frac{g\rho_{01}V_h^2}{V_h^2 - 1}\left(1 - \frac{\gamma}{2} - \frac{i\gamma\Omega}{V_{v1}}\right), \\ G_{(1,2)} &= g\Pi_{11} + \left[iK_{z1} - \frac{1}{2}\right]\frac{\Pi_{21}}{H_1} = \frac{g\rho_{01}V_h^2}{V_h^2 - 1}\left(1 - \frac{\gamma}{2} + \frac{i\gamma\Omega}{V_{v1}}\right), \\ G_{(2,1)} &= g\Pi_{12} - \left[iK_{z2} + \frac{s}{2}\right]\frac{\Pi_{22}}{H_1} = \frac{g\rho_{02}sV_h^2}{sV_h^2 - 1}\left(1 - \frac{\gamma}{2} - \frac{i\gamma\Omega}{sV_{v2}}\right), \end{aligned} \tag{23}$$

with vertical phase velocities:

$$V_{v1} = \frac{\Omega}{K_{z1}} = \frac{V_h\Omega}{\sqrt{V_h^2(\Omega^2 - \Omega_{co}^2) - (\Omega^2 - \Omega_{BV}^2)}}, \tag{24}$$

and

$$V_{v2} = \frac{\Omega}{K_{z2}} = \frac{V_h\Omega}{\sqrt{sV_h^2(\Omega^2 - s\Omega_{co}^2) - (\Omega^2 - s\Omega_{BV}^2)}}. \tag{25}$$

For propagating incident and transmitted gravito-acoustic waves V_{v1}^2 and V_{v2}^2 are positive (meaning K_{z1}, K_{z2} are real and positive), while if $V_{v1}^2, V_{v2}^2 < 0$ the waves are evanescent (meaning K_{z1}, K_{z2} are imaginary). In this paper only propagating waves are analysed. Boundary conditions Eqs. (14) and (15) applied to Eqs. (16), (17), (21), (22) with (23) at $z = 0$, yield the following set of two algebraic equations for complex amplitudes A_r and A_t:

$$\begin{aligned} A_t - A_r &= 1 \\ [G_{(2,1)} - g\rho_{02}]A_t - [G_{(1,2)} - g\rho_{01}]A_r &= G_{(1,1)} - g\rho_{01}, \end{aligned} \tag{26}$$

whose solutions are:

$$A_r = \frac{G_{(1,1)} - G_{(2,1)} + g[\rho_{02} - \rho_{01}]}{G_{(2,1)} - G_{(1,2)} - g[\rho_{02} - \rho_{01}]}, \tag{27}$$

$$A_t = \frac{G_{(1,1)} - G_{(1,2)}}{G_{(2,1)} - G_{(1,2)} - g[\rho_{02} - \rho_{01}]}. \tag{28}$$

These are general equations for the reflection and transmission amplitudes of the gravito-acoustic waves. Notice that $A_t = 2$ if $A_r = 1$. The fact that reflection and transmission amplitudes A_r and A_t turn out to be in general complex indicates that the reflected and transmitted waves are shifted in phase with respect to the incident one.

4. TRANSMISSION COEFFICIENT FOR GRAVITO-ACOUSTIC WAVES

My goal is to derive equation for the transmission coefficient of gravito-acoustic waves, which is defined as the square of the transmission amplitude A_t modulus. Using dimensionless parameters, equation (30) for the transmission amplitude A_t becomes:

$$A_t = \frac{\frac{2\gamma^2\Omega^2}{V_{v1}^2(V_h^2-1)}\left(\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right)}{\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]^2+\frac{\gamma^2\Omega^2}{V_{v1}^2}\left[\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right]^2}$$
$$+i\frac{\frac{2\gamma\Omega}{V_{v1}(V_h^2-1)}\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]}{\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]^2+\frac{\gamma^2\Omega^2}{V_{v1}^2}\left[\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right]^2} \tag{29}$$

It is easy to see that transmission coefficient $T = |A_t|^2$ for gravito-acoustic waves is given by equation:

$$T = \left[\frac{\frac{2\gamma^2\Omega^2}{V_{v1}^2(V_h^2-1)}\left(\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right)}{\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]^2+\frac{\gamma^2\Omega^2}{V_{v1}^2}\left[\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right]^2}\right]^2$$
$$+\left[\frac{\frac{2\gamma\Omega}{V_{v1}(V_h^2-1)}\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]}{\left[\left(1-\frac{\gamma}{2}\right)\left(\frac{1}{V_h^2-1}-\frac{s^2}{sV_h^2-1}\right)+\frac{(s-1)}{V_h^2}\right]^2+\frac{\gamma^2\Omega^2}{V_{v1}^2}\left[\frac{V_{v1}}{V_{v2}}\cdot\frac{s}{sV_h^2-1}+\frac{1}{V_h^2-1}\right]^2}\right]^2. \tag{30}$$

Notice that for $\Omega \gg 1$ and finite values V_h, V_{v1} and V_{v2}, the transmission coefficient of the pure acoustic case could be derived:

$$T = \frac{4(sV_h^2-1)}{\left(s\sqrt{V_h^2-1}+\sqrt{sV_h^2-1}\right)^2}. \tag{31}$$

5. RESULTS AND CONCLUSIONS

1. For $V_h \approx 1$ and $\Omega \gg 1$, there are propagating incident pure acoustic waves, while for $V_h \approx 1/\sqrt{s} \approx 12.9$ and $\Omega \gg 1$, there are propagating transmitted acoustic waves (Fig.2-Fig.3). For pure acoustic waves $V_h = 1/\sqrt{s}$ is a separation point between propagating ($V_h > 1/\sqrt{s}$) and evanescent ($V_h < 1/\sqrt{s}$) transmitted pure acoustic waves. In this point transmission coefficient is equal to zero (Fig.4) and there is no transmitted waves from photosphere to corona. The total waves energy is reflected back to photosphere.
   Eq. (31) shows that there is $T = 4$ for $V_h = 1$ and this is point of separation

between propagating ($V_h > 1$) and evanescent ($V_h < 1$) incident pure acoustic waves.

2. When gravity effects are introduced than we have modified acoustic waves. The conditions for propagating incident modified acoustic waves are $\Omega > \Omega_{co}$ and $V_h > 1$ (derived for $V_{v1} > 0$, Eq.(24)), while for propagating transmitted waves conditions $\Omega > \sqrt{s}\Omega_{co}$ and $V_h > 1/\sqrt{s}$ must be fulfilled ($V_{v2} > 0$, Eq.(25)). This means that for $V_h > 1/\sqrt{s}$ we are dealing with propagating incident and transmitted modified acoustic waves (Fig.2-Fig.3).
3. Fig.5 shows the transmission coefficient T of propagating modified acoustic waves as a function of frequency, while V_h is given parameter. Coefficient $T = 0$ for $\Omega = \Omega_{co}$. For the $\Omega > \Omega_{co}$ transparency from photosphere to corona is smaller for dimensionless horizontal phase velocity $V_h \approx 1/\sqrt{s}$ than for the higher values of V_h. Gravity influence is significant for the frequency range $\Omega_{co} < \Omega < 1$, while for $\Omega \gg 1$ the figure is similar as in the pure acoustic case.
4. The conditions for propagating incident gravity waves are $\Omega < \Omega_{BV}$ and $V_h < \Omega_{BV}/\Omega_{co} \approx 0.98$, and for propagating transmitted gravity waves must be $\Omega < \sqrt{s}\Omega_{BV}$ and $V_h < \Omega_{BV}/\sqrt{s}\Omega_{co} \approx 12.65$. This means that for the $V_h < \Omega_{BV}/\Omega_{co}$ there are propagating incident and transmitted gravity waves (Fig.2-Fig.3).
5. Fig.6 represents the coefficient T of gravity waves. For a small values of dimensionless horizontal phase velocity V_h, transmission coefficient is also small and the waves are mostly reflected back to the photosphere. Transparency is zero for the frequencies given by $\Omega = \sqrt{V_h^2\Omega_{co}^2 - \Omega_{BV}^2/V_h^2 - 1}$. This is a critical frequency because for the frequencies greater than this, incident gravity waves are evanescent. The highest transparency from photosphere to corona have the waves with middle values of dimensionless horizontal phase velocity V_h.

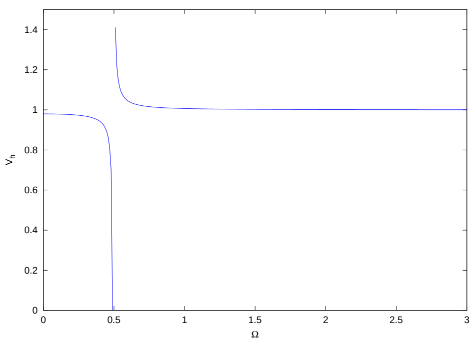

Fig. 2 – Propagating incident acoustic and gravity waves for $s = 0.006$.

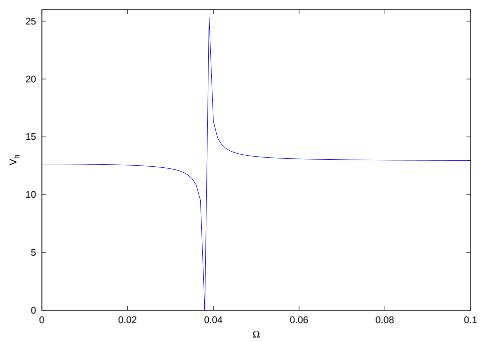

Fig. 3 – Propagating transmitted acoustic and gravity waves for $s = 0.006$.

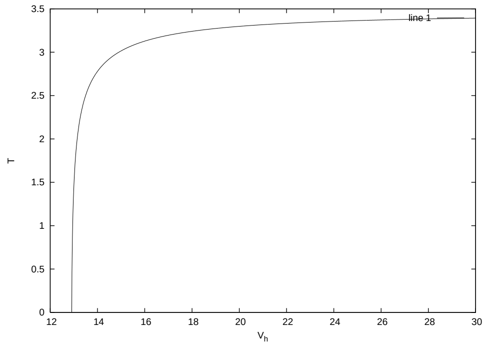

Fig. 4 – The transmission coefficient T for $\Omega \gg 1$.

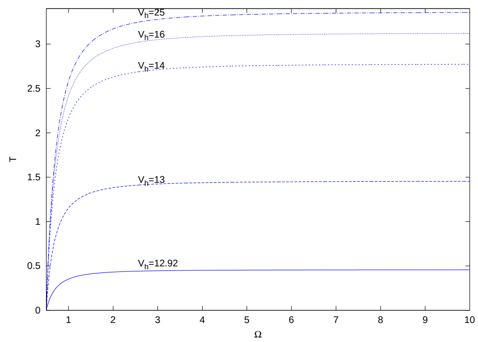

Fig. 5 – The transmission coefficient T of modified acoustic waves.

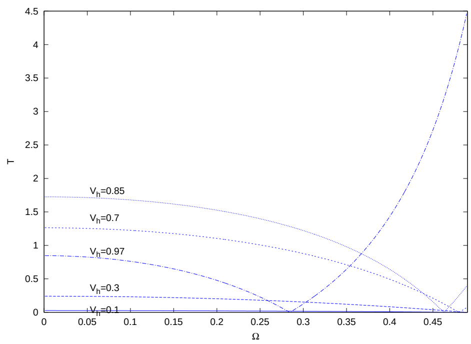

Fig. 6 – The transmission coefficient T of gravity waves.

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