Romanian Reports in Physics, Vol. 65, No. 4, P. 1155–1169, 2013

FIRST INTEGRAL METHOD AND EXACT SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ARISING IN MATHEMATICAL PHYSICS

ZAI-YUN ZHANG[1, a], JUAN ZHONG[1], SHA SHA DOU[1], JIAO LIU[1], DAN PENG[1], TING GAO[1]

[1]School of Mathematics, Hunan Institute of Science and Technology,
College road, Yueyang, 414006, Hunan Province, P.R. China,
E-mail[a]: zhangzaiyun1226@126.com

Received January 20, 2013

Abstract. In this paper, the first integral method (FIM) is employed to solve the perturbed nonlinear Schrödinger's equation (NLSE) with Kerr law nonlinearity, Klein-Gordon-Zakharov equations, Drinfeld-Sokolov system and the perturbed Klein-Gordon equation with local inductance and dissipation effect. By using this method, we obtain the exact travelling wave solutions. It is shown that the method is effective and direct method, based on the ring theory of commutative algebra.

Key words: Exact solutions, NLSE with Kerr law nonlinearity, Klein-Gordon-Zakharov equations (KGZs), Drinfeld-Sokolov system, the perturbed Klein-Gordon equation with local inductance and dissipation effect, the first integral method (FIM).

PACS: 05.45.Yv, 02.30.Jr, 42.81.Dp

1. INTRODUCTION

It is well known that travelling wave solutions of Nonlinear Partial Differential Equations (NPDEs) play an important role in the study of nonlinear wave phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibres, etc. (see for instance Ref. [1]). In the recent decade, several methods for finding the exact solutions to NPDEs have been proposed, such as the trigonometric function series method [1, 2], the modified mapping method and the extended mapping method [3], the modified trigonometric function series method [4, 5], the dynamical system approach and the bifurcation method [6, 7, 15], the infinite series method and Jacobi elliptic function expansion method [8], the $\exp$-function method [9, 16], the multiple $\exp$-function method [10], the transformed rational function method [11], the symmetry algebra method (consisting of Lie point symmetries) [12], the Wronskian technique [13], the linear superposition principle [14], the improved Fan subequation method [17], $(\frac{G'}{G})$-expansion method [18-22], the modified $(\frac{G'}{G})$-method [23], Multiple $(\frac{G'}{G})$-expansion method [24], Lie classical approach and (G'/G)-expansion method [25], Lie transform perturbation method

[26], the trial equation method [27], the homogeneous balance method [28], the modified $\tanh-\coth$ function method [29], the transformed rational function method [30], the auxiliary equation method [31], the auxiliary equation method [32], and so on.

In the pioneering work, applying the theory of commutative algebra, Feng [33] proposed a new approach which is currently called the first integral method for a reliable treatment of the NPDEs. The useful first integral method is widely used in many contributions such as in Refs. [34-38] and the references therein. The method is reliable, effective, precise and does not require complicated and tedious computations. The main idea of the first integral method is to find first integrals of nonlinear differential equations in polynomial form. Taking the polynomials with unknown polynomial coefficients into account, the method provides exact and explicit solutions.

2. DESCRIPTION OF THE FIRST INTEGRAL METHOD (FIM)

In this section, to facilitate further on our analysis, we initiate our study by briefly reviewing the procedure.

Step 1. Consider a general nonlinear PDE in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{1}$$

where P is a polynomial in its arguments and subscripts denote partial derivatives. By means of the transformation $u = U(\xi)$, $\xi = kx + \omega t + \xi_0$, we reduce Eq. (1) to an ordinary differential equation (ODE) of the form

$$P(U, kU', \omega U', k^2U'', k\omega U'', \omega^2 U'', \cdots) = 0, \tag{2}$$

where k, ω and ξ_0 are arbitrary constants, $U = U(\xi)$ and the primes denote ordinary derivatives with respect to ξ.

Step 2. Next, we introduce a new independent variable

$$x(\xi) = u(\xi), y = \frac{\partial u(\xi)}{\partial \xi}, \tag{3}$$

which leads to the system of nonlinear ordinary differential equations (ODE)

$$\frac{\partial x(\xi)}{\partial \xi} = y(\xi), \frac{\partial y(\xi)}{\partial \xi} = F_1(x(\xi), y(\xi)). \tag{4}$$

Step 3. Based on the qualitative theory of differential equations [39], if one can find the first integrals to System (4) under the same conditions, the analytic solutions to (4) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. A key idea of this approach here to find the first integral is to utilize the Division Theorem. We will apply the Division Theorem to obtain one first integral to (4) which reduces (2) to a first order integrable ODE. For convenience, first let us recall the Division Theorem for two variables in the complex domain C [40].

Division Theorem: Suppose that $P(w,z)$, $Q(w,z)$ are polynomials in $C(w,z)$ and $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C(w,z)$ such that $Q(w,z) = P(w,z)G(w,z)$.

3. APPLICATIONS

In this section, we discuss two problems involving the nonlinear PDEs by using the first integral method described in Sect. 2.

Example 1. We first begin with the nonlinear equation of the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity given in Ref.[3]

$$iu_t + u_{xx} + \alpha|u|^2u + i[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u|^2)_x u] = 0. \quad (5)$$

where γ_1 is the third order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is a also a version of nonlinear dispersion. It must be very clear that γ_3 is not Raman Scattering. It is only when γ_3 is purely imaginary, it is Raman scattering. Moreover, Raman scattering is not a Hamiltonian perturbation and therefore it is not an integrable perturbation. More details are presented in Ref.[6]. Eq.(5) describes the propagation of optical solitons in nonlinear optical fibres that exhibits a Kerr law nonlinearity, we can see Ref.[4].

Assume that Eq.(5) has travelling wave solutions in the form [3]

$$u(x,t) = \phi(\xi)\exp(i(Kx - \Omega t)),\ \xi = k(x - ct), \quad (6)$$

where c is the propagation speed of a wave.

Substituting (6) into Eq.(5) yields

$$\begin{aligned} &i(\gamma_1 k^3\phi''' - 3\gamma_1 K^2 k\phi' + \gamma_2 k\phi^2\phi' + 2\gamma_3 k\phi^2\phi' - ck\phi' + 2Kk\phi') \\ &\qquad + (\Omega\phi + k^2\phi'' - K^2\phi + \alpha\phi^3 + 3\gamma_1 Kk^2\phi'' + \gamma_1 K^3\phi - \gamma_2 K\phi^3) = 0, \end{aligned}$$

where $\gamma_i (i = 1,2,3)$, α, k are positive constants and the prime meaning differentiation with respect to ξ.

By virtue of Ref. [2, pp. 3065], we have

$$\gamma_1 k^2\phi''(\xi) + (2K - c - 3\gamma_1 K^2)\phi(\xi) + (\frac{1}{3}\gamma_2 + \frac{2}{3}\gamma_3)\phi^3(\xi) = 0.$$

That is,

$$\phi''(\xi) - A\phi(\xi) + B\phi^3(\xi) = 0, \quad (7)$$

where $A = -\frac{2K-c-3\gamma_1 K^2}{\gamma_1 k^2}$, $B = \frac{\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3}{\gamma_1 k^2}$.

By virtue of (4), we get

$$\frac{\partial x(\xi)}{\partial \xi} = y(\xi), \frac{\partial y(\xi)}{\partial \xi} = Ax(\xi) - Bx^3(\xi). \tag{8}$$

According to the FIM, we suppose that we suppose the $x(\xi)$ and $y(\xi)$ are the nontrivial solutions of (4) and

$$Q(x,y) = \sum_{i=0}^{N} a_i(x) y^i = 0$$

is an irreducible polynomial in the complex domain $C(x,y)$, such that

$$Q(x(\xi), y(\xi)) = \sum_{i=0}^{N} a_i(x(\xi)) y^i(\xi) = 0, \tag{9}$$

where $a_i(x)(i = 0,1,\cdots,N)$ are polynomials of x and $a_N(x) \neq 0$. Eq.(9) is called the first integral to (8). Owing to the Division Theorem, there exists a polynomial $g(x)+h(x)y$ n the complex domain in the complex domain $C(x,y)$, such that

$$\frac{\partial Q}{\partial \xi} = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial \xi} = (g(x)+h(x)y)\sum_{i=0}^{N} a_i(x(\xi)) y^i(\xi) = 0. \tag{10}$$

In this example, we take two different cases, assuming that $N = 1$ and $N = 2$ in (9), we have

Case 1. Assume that $N = 1$, by comparing with the coefficients of $y^i(i = 0,1)$ from both sides of (10), we conclude

$$\frac{\partial a_1(x)}{\partial \xi} = h(x) a_1(x), \tag{11}$$

$$\frac{\partial a_0(x)}{\partial \xi} = g(x) a_1(x) + h(x) a_0(x), \tag{12}$$

$$a_1(x)(Ax - Bx^3) = g(x) a_0(x). \tag{13}$$

Since $a_i(x)(i = 0,1)$ are polynomials, then from (11) we deduce that $a_1(x)$ is constant and $h(x) = 0$. For simplicity, taking $a_1(x) = 1$, and balancing the degrees of $g(x)$, $a_1(x)$ and $a_0(x)$, we conclude that $deg(g(x)) = 1$. Suppose that $g(x) = A_1 x + B_0$, then from (12) we find $a_0(x)$ as follows:

$$a_0(x) = B_1 + B_0 x + \frac{1}{2} A_1 x^2, \tag{14}$$

where A_1, B_0 are arbitrary integration constant, and B_1 is arbitrary integration constant to be determined.

Substituting $g(x)$, $a_1(x)$ and $a_0(x)$ into (13) and setting all the coefficients of x to be zero, then we obtain a system of nonlinear equations and by solving it, we deduce

$$A_1 = \sqrt{-2B}, B_0 = 0, B_1 = \frac{A}{\sqrt{-2B}}. \tag{15}$$

$$A_1 = -\sqrt{-2B}, B_0 = 0, B_1 = -\frac{A}{\sqrt{-2B}}. \tag{16}$$

It follows from (9) and (15) that

$$y = \sqrt{\frac{-B}{2}}(\frac{A}{B} - x^2). \tag{17}$$

Using (4) and (8), we obtain the exact solution to equation (8) and the exact travelling solution to the NLSE with Kerr law nonlinearity as follows

$$u_1 = \sqrt{\frac{A}{B}} tan[\sqrt{\frac{A}{B}}(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0] \exp(i(Kx - \Omega t))$$

and then the exact travelling solution to Eq.(5) can be written as

$$|u_1(x,t)| = |\sqrt{\frac{A}{B}}\tan[\sqrt{\frac{A}{B}}(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \tag{18}$$

where ξ_0 are arbitrary constants, $|u|$ is the norm of u. Similarly, in the case of (16), from (9), we obtain

$$y = \sqrt{\frac{-B}{2}}(x^2 - \frac{A}{B}),$$

then the exact travelling solution to Eq.(5) can be written as

$$|u_2(x,t)| = |\sqrt{\frac{A}{B}}\coth[\sqrt{\frac{A}{B}}(-\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|. \tag{19}$$

Case 2. Assume that $N = 2$, by comparing with the coefficients of $y^i (i = 0,1)$ from both sides of (10), we have

$$\frac{\partial a_2(x)}{\partial \xi} = h(x)a_2(x), \tag{20}$$

$$\frac{\partial a_1(x)}{\partial \xi} = g(x)a_2(x) + h(x)a_1(x), \tag{21}$$

$$\frac{\partial a_0(x)}{\partial \xi} = -2a_2(x)(Ax - Bx^3) + g(x)a_2(x) + h(x)a_0(x), \tag{22}$$

$$a_1(x)(Ax - Bx^3) = g(x)a_0(x). \tag{23}$$

Since $a_i(x)$ $(i = 0,1,2)$ are polynomials, then from (10) we deduce that $a_2(x)$ is constant and $h(x) = 0$. For simplicity, taking $a_2(x) = 1$, and balancing the degrees of $g(x)$, $a_1(x)$ and $a_0(x)$, we conclude that $deg(g(x)) = 1$ only. Suppose that $g(x) = A_1x + B_0$, then from (21) and (22) we find $a_1(x)$ and $a_0(x)$ as follows:

$$a_1(x) = B_1 + B_0x + \frac{1}{2}A_1x^2, \tag{24}$$

$$a_0(x) = d + B_1B_0x + \frac{1}{2}(B_0^2 - 2A + B_1A_1)A_1x^2 + \frac{1}{2}A_1B_0x^3 + \frac{1}{4}(2B + \frac{1}{2}A_1^2)x^4, \tag{25}$$

where A_1, B_0 are arbitrary integration constants, and B_1, d are arbitrary integration constants to be determined.

Substituting $g(x)$, $a_1(x)$ and $a_0(x)$ into (23) and setting all the coefficients of x to be zero, then we obtain a system of nonlinear equations and by solving it, we deduce

$$A_1 = 2\sqrt{-2B}, B_0 = 0, B_1 = -\frac{A}{\sqrt{-2B}}, d = -\frac{A^2}{2B}, \tag{26}$$

$$A_1 = 2\sqrt{-2B}, B_0 = 0, B_1 = \frac{A}{\sqrt{-2B}}, d = -\frac{A^2}{2B}. \tag{27}$$

It follows from (10), (26) and (27) that

$$y = \pm\sqrt{\frac{-B}{2}}(x^2 - \frac{A}{B}). \tag{28}$$

Combining (28) with (8), we obtain the following exact travelling solution to Eq.(5)

$$|u_3(x,t)| = |\sqrt{\frac{A}{B}}tan[(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \tag{29}$$

where ξ_0 are arbitrary constants, $|u|$ is the norm of u.

Example 2. We consider the Klein-Gordon-Zakharov given in Ref.[5] or [41]

$$u_{tt} - u_{xx} + u + \alpha nu = 0, \tag{30}$$

$$n_{tt} - n_{xx} = \beta(|u|^2)_{xx}, \tag{31}$$

where function $u(x,t)$ denotes the fast time scale component of electric field raised by electrons and the function $n(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function, $n(x,t)$ is a real function, α, β are two nonzero real parameters. This system describes the interaction of the Langmuir wave and the ion acoustic wave in a high frequency plasma. More details are presented in Ref.[41] and the references therein. Based on the modified trigonometric function series method (MTFSM) [4], Zhang *et al.* [5] studied the travelling wave solutions of KGZEs (30) and (31). More precisely, we combined the trigonometric function series method with the exp-function method. More details are presented in Ref.[5]. Quite recently, Zhang *et al.* [41] investigated the bifurcations and dynamic behaviour of travelling wave solutions to the Klein-Gordon-Zakharov equations given in Ref.[5]. Under different parameter conditions, we obtained some exact explicit parametric representations of travelling wave solutions by using the bifurcation method and phase plane analysis technique [6, 7].

To facilitate further on our analysis, we assume that Eq.(30) has travelling wave solutions in the form [5, 41]

$$u(x,t) = \phi(x,t)\exp(i(kx+\omega t+\xi_0)), \tag{32}$$

where $u(x,t)$ is a real-valued function, k, ω are two real constants to be determined, ξ_0 is an arbitrary constant. Substituting (32) into (30) and (31) yields

$$\phi_{tt} - \phi_{xx} + (k^2 - \omega^2 + 1)\phi + \alpha n\phi = 0, \tag{33}$$

$$\omega\phi_t - k\phi_x = 0, \tag{34}$$

$$n_{tt} - n_{xx} = \beta(\phi^2)_{xx}. \tag{35}$$

By virtue of (34), we assume

$$\phi(x,t) = \phi(\xi) = \phi(\omega x + kt + \xi_1), \tag{36}$$

where ξ_1 is an arbitrary constant. Substituting (36) into (33), we have

$$n(x,t) = \frac{(\omega^2 - k^2)\phi''(\xi)}{\alpha\phi(\xi)} + \frac{(\omega^2 - 1 - k^2)}{\alpha}. \tag{37}$$

Hence, we can also assume

$$n(x,t) = \psi(\xi) = \psi(\omega x + kt + \xi_1). \tag{38}$$

Substituting (38) into (35) and integrating the resultant equation twice with respect to ξ, we obtain

$$\psi(\xi) = \frac{\beta\omega^2\phi^2(\xi)}{k^2 - \omega^2} + C, \tag{39}$$

where C is an integration constant. For simplicity, we choose $C = 0$. It follows from (33) and (39) that

$$\phi''(\xi) + \frac{k^2 - \omega^2 + 1}{k^2 - \omega^2}\phi(\xi) + \frac{\alpha\beta\omega^2}{(k^2 - \omega^2)^2}\phi^3(\xi) = 0. \tag{40}$$

For simplicity, we assume $A = -\frac{k^2-\omega^2+1}{k^2-\omega^2}$, $B = \frac{\alpha\beta\omega^2}{(k^2-\omega^2)^2}$, thus (40) yields to ordinary differential equation(ODE) (7) as follows:

$$\phi''(\xi) - A\phi(\xi) + B\phi^3(\xi) = 0.$$

Based on FIM in section 2 and example 1 in section 3, it is easy to obtain the travelling wave solutions to the KGZs (30) and (31) as follows:

$$\begin{cases} |u_1(x,t)| = |\sqrt{\frac{A}{B}}tan[\sqrt{\frac{A}{B}}(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \\ |n_1(x,t)| = |\frac{\beta\omega^2}{k^2-\omega^2}\sqrt{\frac{A}{B}}tan[\sqrt{\frac{A}{B}}(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \end{cases} \tag{41}$$

$$\begin{cases} |u_2(x,t)| = |\sqrt{\frac{A}{B}}coth[\sqrt{\frac{A}{B}}(-\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \\ |n_2(x,t)| = |\frac{\beta\omega^2}{k^2-\omega^2}\sqrt{\frac{A}{B}}coth[\sqrt{\frac{A}{B}}(-\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \end{cases} \tag{42}$$

$$\begin{cases} |u_3(x,t)| = |\sqrt{\frac{A}{B}}tan[(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|, \\ |n_3(x,t)| = |\frac{\beta\omega^2}{k^2-\omega^2}\sqrt{\frac{A}{B}}tan[(\sqrt{\frac{-B}{2}}k(x-ct)) + \xi_0]|. \end{cases} \tag{43}$$

Remark 3.1. Indeed, Eq.(7) is the well known the Duffing equation. It is well known that the Duffing equation is the equation governing the oscillations of a mass attached to the end of a spring whose tension (or compression). We can see the Ref.[42].

Example 3. We investigate the nonlinear Drinfeld-Sokolov system given in Ref.[43]

$$\begin{cases} u_t + v_x^2 = 0, \\ v_t - v_{xxx} + (3uv)_x = 0. \end{cases} \tag{44}$$

Introducing the following transformations

$$u(x,t) = f(\xi), v(x,t) = g(\xi), \tag{45}$$

where $\xi = x - ct$, the system (45) yields

$$-c\frac{df}{d\xi} + \frac{dg^2}{d\xi} = 0, \tag{46}$$

$$-c\frac{dg}{d\xi} - \frac{d^3g}{d\xi^3} + \frac{d(3fg)}{d\xi} = 0. \tag{47}$$

Integrating equation (47), we obtain $f(\xi)$ as

$$f(\xi) = \frac{g^2 - \alpha}{c}, \tag{48}$$

where α is an arbitrary integration constant (assume $\alpha = 0$). Substituting $f(\xi)$ into equation (47) and setting the integration constant to be zero, we arrive at

$$\frac{d^2g}{d\xi^2} + cg - \frac{3}{c}g^3 = 0.$$

Assume that $A = -c$, $B = -\frac{3}{c}$, then the above equation becomes Duffing type equation (7)

$$g''(\xi) - Ag(\xi) + Bg^3(\xi) = 0. \tag{49}$$

Based on FIM in section 2 and example 1 in section 3 and noticing (48), (49), it is easy for us to obtain the travelling wave solutions to the nonlinear Drinfeld-Sokolov system (44). Here, we omit the procedure.

Example 4. We study the perturbed Klein-Gordon equation with cubic nonlinearity in (1+1)-Dimension with local inductance and dissipation effect as follows

$$u_{tt}-u_{xx}+f(u)=\varepsilon(\alpha u+pu_t+qu_x+\beta u_{xt}+\gamma u_{tt}), \tag{50}$$

where $f(u)=au-bu^3$, a, b, α, p, q, β, γ are constants and ε is the perturbation parameter.

Recently, Sassaman and Biswas [44, 45] investigated the perturbed KGE (50) and obtained the exact 1-soliton solution. These perturbation terms typically arise in the study of long Josephson junction in the context of sine-Gordon equation (SGE). Since SGE can be approximated by KGE, an exact solution of the perturbed KGE will make sense in the context of the study of the SGE. For the perturbation terms, α represents losses across the junction, p accounts for dissipative losses in Josephson junction theory due to tunnelling of normal electrons across the dielectric barrier, q is generated by a small inhomogeneous part of the local inductance, β represents diffusion and γ is the capacity inhomogeneity. More details are presented in Ref.[45]. In fact, the term pu_t (call dissipation effect) is generated by a variety of dissipative mechanisms. As for the dissipation effect, we can see the Ref.[46, 47].

When $\varepsilon=0$, Eq.(50) reduces to KGE

$$u_{tt}-u_{xx}+\alpha u-\beta u^3=0. \tag{51}$$

Applying trigonometric function series method, Zhang [1] studied the Eq.(51) and obtained the new exact travelling wave solutions are complex linear combinations of kink solitary wave solutions and bell solitary wave solutions. Eq.(51) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a “splay wave” along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc. More details are presented in Ref.[1] and the references therein. Quite recently, based on modified G'/G-expansion method [23], Xiao and Zhang [18] investigated the Eq.(51) and obtained the exact travelling wave solutions are expressed in terms of hyperbolic functions, the trigonometric functions and the rational functions.

In absence of perturbed term, there are many researchers investigated the exact travelling wave solutions of KGE and generalized Klein-Gordon equation (gKGE), such as Ref.[48-53]. In Ref.[50], Sassaman and Biswas obtained the exact 1-soliton solution of five different forms of the gKGE by using the solitary wave solution ansatz. In Ref.[51], Sassaman and Biswas obtained the exact 1-soliton solution of five different forms of the KGE in 1 + 2 dimensions. In Ref.[52], Sassaman and Biswas investigated the coupled Klein-Gordon equations in (1+1) and (1+2) dimensions with cubic law of nonlinearity and arbitrary power law nonlinearity. Then, they obtained the 1-soliton solution of the coupled system. In Ref.[53], Sassaman *et al.* studied topological and non-topological soliton solutions of five different forms of the gKGE in 1+2 dimensions. However, in presence of perturbed term, we can see Ref.[54-57]. In Ref.[54], Sassaman and Biswas obtained the 1-soliton solution to the perturbed KGE, by He's semi-inverse variational principle. In Ref.[55], Sassaman, Heidari and Biswas obtained the topological and non-topological soliton solutions of the perturbed KGE, by the He's semi-inverse variational principle and carried out the integration of the KGE with five types of nonlinearity in presence of a few perturbation terms. In Ref.[56], Sassaman and Biswas obtained the adiabatic variation of the soliton velocity, in presence of perturbation terms, of the phi-four model and the nonlinear Klein-Gordon equations. There are three types of models of the nonlinear Klein-Gordon equation, with power law nonlinearity. In Ref.[57], Esfahani considered the solitary wave solutions of the perturbed KGE by using the sech-ansatz method.

Remark 3.2. We notice that the perturbation terms considered in the context of Eq.(50) are relevant to perturbation of sine-Gordon equation in the context of long Josephson junction. In Eq.(50), for the perturbation terms, α represents losses across the junction, p accounts for dissipative losses in Josephson junction theory due to tunnelling of normal electrons across the dielectric barrier, q is generated by a small inhomogeneous part of the local inductance, β represents diffusion and γ is the capacity inhomogeneity. More details are present in Ref.[58]. For soliton perturbation in the other physical model, including generalized Klein-Gordon equation with full nonlinearity, the Bretherton equation, Ito equation, the modified complex Ginzburg Landau equation, Benney-Luke equation, DWDM systems, the generalized nonlinear Schrodinger's equation, Boussinesq-Burgers equation *et al.* we can see Ref.[59-70].

To facilitate further on our analysis, we assume that Eq.(50) has travelling wave solutions in the form

$$u(x,t) = u(\xi), \xi = x - \omega t, \tag{52}$$

where ω is the propagation speed of a wave.

Substituting (52) into Eq.(50) yields

$$(\omega^2 - 1 + \varepsilon\beta\omega - \varepsilon\gamma\omega^2)u'' + (\varepsilon q\omega - \varepsilon p)u' + (a - \varepsilon\alpha)u - bu^3 = 0,$$

where $u' = u_\xi$, $u'' = u_{\xi\xi}$.

Assume that

$$A = -\frac{\varepsilon q\omega - \varepsilon p}{\omega^2 - 1 + \varepsilon\beta\omega - \varepsilon\gamma\omega^2}, B = -\frac{a - \varepsilon\alpha}{\omega^2 - 1 + \varepsilon\beta\omega - \varepsilon\gamma\omega^2}, C = \frac{b}{\omega^2 - 1 + \varepsilon\beta\omega - \varepsilon\gamma\omega^2},$$

so the above equation is transformed into the following form:

$$u''(\xi) - Au'(\xi) - Bu(\xi) - Cu^3(\xi) = 0, \tag{53}$$

where A, B, C are constants.

By FIM, Eq.(53) is transformed to the following system of ODEs

$$\frac{\partial x(\xi)}{\partial \xi} = y(\xi), \frac{\partial y(\xi)}{\partial \xi} = Ay(\xi) + Bx(\xi) + Cx^3(\xi). \tag{54}$$

Now, we apply the Division Theorem to look for the first integral to (54). Suppose that $x(\xi)$ and $y(\xi)$ are the nontrivial solutions to (54), and

$$P(x,y) = \sum_{j=0}^{m} a_j(x) y^j = 0, \tag{55}$$

where $a_j(x)(j = 0,1,\cdots,m)$ are polynomials of X and all relatively prime in $C(x,y)$, $a_m(x) \neq 0$. Eq.(55) is also called the first integral of (3.49). Note that $P(x,y)$ is a polynomial in x and y and $\frac{dP}{d\xi}$ implies $\frac{dP}{d\xi} = 0$. By the Division Theorem, there exists a polynomial $H(x,y) = (g(x) + h(x)y)$ in the complex domain $C(x,y)$, such that

$$\frac{dP}{d\xi} = \frac{dP}{dx}\frac{dx}{d\xi} + \frac{dP}{dy}\frac{dy}{d\xi} = (h(x) + g(x)y) \sum_{i=0}^{N} a_i(x(\xi)) y^i(\xi) = 0. \tag{56}$$

On equating the coefficients of $y_j (j = 0,1,2,3)$ on both sides of equation (56), we obtain

$$\mathbf{a}'(x) = \mathbf{A}(x) \cdot \mathbf{a}(x) \tag{57}$$

and

$$[0, Bx + Cx^3, -g(x)] \cdot \mathbf{a}(x) = 0, \tag{58}$$

where $\mathbf{a}(x) = (a_2(x), a_1(x), a_0(x))^T$, and

$$\mathbf{A}(x) = \begin{pmatrix} h(x) & 0 & 0 \\ g(x) - 2A & h(x) & 0 \\ -2(Bx + Cx^3) & g(x) - A & h(x) \end{pmatrix}.$$

Since $a_j(x)(j = 0,1,2)$ are polynomials, from (57) we deduce that $a_2(x)$ is a constant and $h(x) = 0$. For simplification, taking $a_2(x) = 1$ and solving (57), we have

$$\mathbf{a}(x) = \begin{pmatrix} 1 \\ \int (g(x) - 2A) dx \\ \int [a_1(x) g(x) - A a_1(x) - 2(Bx + Cx^3)] dx \end{pmatrix}. \tag{59}$$

From (58) and (59), we conclude that either $deg g(x) = 0$ or $deg g(x) = 1$, *i.e.* either $deg a_1(x) = 1$ or $deg a_1(x) = 2$. Otherwise, if $deg g(x) = k > 1$, then we deduce $deg a_1(x) = k+1$ and $deg a_0(x) = 2k+2$ from (59). This yields a contradiction with (58), since the degree of the polynomial $a_1(x) \cdot (Bx + Cx^3)$ is $k+4$, but the degree of the polynomial $a_0(x) \cdot g(x)$ is $3k+2$.

In case $degg(x)=0$, assume $g(x)=g(g\in\mathcal{C})$ and $a_1(x)=A_1x+A_0$, $A_1,A_0\in\mathcal{C}$ with $A_1\neq 0$. From (3.55), we get that $A_1=g-2A$ and

$$a_0(x)=-\frac{C}{2}x^4-Bx^2+\frac{A_1(A_1+2A)}{2}x^2+A_0(A_1+2A)x-\frac{AA_1}{2}x^2-A_0A_1x+D,$$

here D is an arbitrary integration constant (assume $D=0$). Substituting $a_1(x)$ and $a_0(x)$ into (58) and setting all coefficients of $x^i(i=4,2,1,0)$ to zero, we have

$$\begin{cases} -2A_1C=(A_1+2A)C, \\ A_0C=0, \\ A_1B=(-B+\frac{A_1g}{2}-\frac{A_1A}{2})\frac{4A}{3}, \\ A_0B=(A_0g-A_0A)\frac{4A}{3}. \end{cases} \tag{60}$$

Simplifying (60), we obtain that $A_1=-\frac{2A}{3}$, $A_0=0$. Then, (60) becomes

$$y^2-\frac{2A}{3}xy-\frac{C}{2}x^4-Bx^2+\frac{A_1(A_1+2A)}{2}x^2+A_0(A_1+2A)x-\frac{AA_1}{2}x^2-A_0A_1x=0. \tag{61}$$

From (61), y can be expressed in terms of x. So, (61) can be factorized as

$$(y+F_1x^2+F_2x+F_3)(y+H_1x^2+H_2x+H_3)=0. \tag{62}$$

If let $y+F_1x^2+F_2x+F_3=0$ or $y+H_1x^2+H_2x+H_3=0$, this coincides with (55) in the case of $m=1$.

When $\Delta=F_2^2-4F_1F_3>0$ (or $\Delta=H_2^2-4H_1H_3>0$), combining (54) with (61) we obtain that Eq. (50) has the exact solution of the form

$$u(x,t)=\frac{B_1C_0e^{B_2(\xi+\xi_0)}+B_3}{1+C_0e^{B_2(\xi+\xi_0)}} \tag{63}$$

where $\xi=x-ct$, $B_i(i=1,2,3)$ are real numbers which depend on F_i (or H_i), and C_0 is an arbitrary integration constant andξ_0 is arbitrary real.

Similarly, when $\Delta=F_2^2-4F_1F_3<0$ (or $\Delta=H_2^2-4H_1H_3<0$), then Eq. (50) has the exact solution of the form

$$u(x,t)=K_1tan[K_2(\xi+\xi_0)+C_0]+K_3, \tag{64}$$

where $\xi=x-ct$, $K_i(i=1,2,3)$ are real numbers which depend on F_i (or H_i), and C_0 is an arbitrary integration constant and ξ_0 is arbitrary real.

When $\Delta=F_2^2-4F_1F_3=0$ (or $\Delta=H_2^2-4H_1H_3=0$), then Eq. (50) has the exact solution of the form

$$u(x,t)=\frac{1}{R_1(\xi+\xi_0+C_0)}+R_2, \tag{65}$$

where $\xi=x-ct$, $R_1=-F_1($ or $R_1=-H_1)$, R_2 is a real number which depends on F_i (or H_i), and C_0 is an arbitrary integration constant and ξ_0 is arbitrary real.

In case $degg(x) = 1$, the argument is identical, so we omit it here.

Remark 3.3. If assuming $m = 3, 4$ in (55), respectively, using the similar arguments as earlier we obtain that (54) does not have any first integral in the form (55). We have no need of discussion for the cases $m \geq 5$ due to the fact that the polynomial equation with the degree greater than or equal to 5 is generally not solvable.

4. CONCLUSION AND DISCUSSION

The first integral method described herein is not only efficient but also has the merit of being widely applicable. The first integral method is applied successfully for solving the system of nonlinear partial differential equations which are the perturbed nonlinear Schrödinger's equation (NLSE) with Kerr law nonlinearity, Klein-Gordon-Zakharov equations, Drinfeld-Sokolov system and the perturbed Klein-Gordon equation with local inductance and dissipation effect. Thus, we deduced that the proposed method can be extended to solve many systems of nonlinear partial differential equations which are arising in mathematical physics. The exact solution of the general system of nonlinear partial differential equations using the first integral method is still an open point of research.

Acknowledgements. The authors would like to present our sincere thanks to the referee for their valuable and helpful comments and suggestions. This work was supported by Hunan Province College Students Research Learning and Innovation Experimental Program (P.R. China).

REFERENCES

1. Z.Y. Zhang, Turk. J. Phys. **32**, 235 (2008).
2. W.X. Ma, B. Fuchssteiner, In. J. Nonlinear Mechanics **31**, 329 (1996).
3. Z.Y. Zhang, Z.H. Liu, X.J. Miao, Y.Z. Chen, Appl. Math. Comput. **216**, 3064 (2010).
4. Z.Y. Zhang, Y.X. Li, Z.H. Liu, X.J. Miao, Commun. Nonlinear Sci. Numer. Simulat. **16**, 3097 (2011).
5. Z.Y. Zhang, Y.H. Zhang, X.Y. Gan, D.M. Yu, Z. Naturforsch. **67a**, 167 (2012).
6. Z.Y. Zhang, Z.H. Liu, X.J. Miao, Y.Z. Chen, Phys. Lett. A, **375**, 1275 (2011).
7. Z.Y. Zhang, X.Y. Gan, D.M. Yu, Z. Naturfors. **66a**, 721 (2011).
8. Z.Y. Zhang, X.Y. Gan, D.M. Yu, Y.H. Zhang, X.P. Li, Commun. Theor. Phys. **57**, 764 (2012).
9. A. Yıldırım and Z. Pınar, Comput. Math. Appl. **60**, 1873 (2010).
10. W.X. Ma, T.W. Huang, Y. Zhang, Phys. Scr. **82**, 065003 (2010).
11. W.X. Ma, J.-H. Lee, Chaos Solitons and Fractals, **42**, 1356 (2009).
12. W.X. Ma, M. Chen, Appl. Math. Comput. **215**, 2835 (2009).
13. W.X. Ma, Y. You, Trans. Amer. Math. Soc. **357**, 1753 (2005).
14. W.X. Ma, E.G. Fan, Comput. Math. Appl. **61**, 950 (2011).
15. A.Y. Chen, Z.J. Ma, Pramana **71**, 57 (2008).
16. J. Lee, R. Sakthivel, Pramana **61**, 565 (2010).

17. Y.A. Xie, S.Q. Tang, D.H. Feng, Pramana **78**, 499 (2012).
18. C.E. Xiao, Z.Y. Zhang, Far East Journal of Applied Mathematics **60**, 131 (2011).
19. H. Kheiri, M.R. Moghaddam, V. Vafaei, Pramana **76**, 831 (2011).
20. B.S. Bahrami, H. Abdollahzadeh, I.M. Berijani, D.D. Ganji, M. Abdollahzadeh, Pramana **77**, 263 (2011).
21. A. Malik, F. Chand, H. Kumar, S.C. Mishra, Pramana **78**, 513 (2012).
22. M. Labidi, G. Ebadi, E. Zerrad, A. Biswas, Pramana **78**, 59 (2012).
23. X.J. Miao, Z.Y. Zhang, Commun. Nonlinear Sci. Numer. Simulat. **16**, 4259 (2011).
24. J.C. Chen, B. Li, Pramana **78**, 375 (2012).
25. S. Kumar, K. Singh, R.K. Gupta, Pramana **79**, 41 (2012).
26. A. Nanayakkara, Pramana **76**, 37 (2011).
27. Y. Gurefe, A. Sonmezoglu, E. Misirli, Pramana **77**, 1023 (2011).
28. M.A. Abdulwahhab, A.H. Bokhari, A.H. Kara, F.D. Zaman, Pramana **77**, 407 (2011).
29. R. Kumar, R.S. Kaushal, A. Prasad, Pramana **75**, 607 (2010).
30. J. Lee, R. Sakthivel, Pramana **76**, 819 (2011).
31. S. Zhang, H.Q. Zhang, Pramana **76**, 561 (2011).
32. H. Li, J.L. Zhang, Pramana **72**, 915 (2009).
33. Z.S. Feng, J. Phys. A: Math. Gen. **35**, 343 (2002).
34. Z.S. Feng, X.H. Wang, Phys. Scr. **64**, 7 (2001).
35. Z.S. Feng, Phys. Lett. A **293**, 57-66 (2002).
36. Z.S. Feng, K. Roger, J. Math. Anal. Appl. **328**, 1435 (2007).
37. Z.S. Feng, Phys. Lett. A **302**, 64 (2002).
38. Z.S. Feng, X.H. Wang, Phys. Lett. A **308**, 173 (2003).
39. T.R. Ding, C.Z. Li, *Ordinary differential equations* (Peking University Press, Peking, 1996).
40. Z.S. Feng, Chaos Soliton Fractals **38**, 481 (2008).
41. Z.Y. Zhang, F.L. Xia, X.P. Li, Pramana **80**(1), 41 (2013).
42. Z.S. Feng, Phys. Lett. A **317**, 115 (2003).
43. A.M. Wazwaz, Commun. Non. Sci. Numer. Simul. **11**, 311 (2006).
44. A. Biswas, C. Zony, E. Zerrad, Appl. Math. Comput. **203**, 153 (2008).
45. R. Sassaman, A. Biswas, Physics Express **1**, 9 (2011).
46. Z.Y. Zhang, X.J. Miao, Comput. Math. Appl. **59**, 1003 (2010).
47. Z.Y. Zhang, Z.H. Liu, X.J. Miao, Y.Z. Chen, J. Math. Phys. **52**, 023502 (2011).
48. M.A. Aguero, M.L. Najera, J.A. Aguilar, J. Sanchez, Int. J. Theor. Phys. **48**, 2098 (2009).
49. K.C. Basak, P.C. Ray, R.K. Bera, Commun. Nonlinear Sci. Numer. Simulat. **14**, 718 (2009).
50. R. Sassaman, A. Biswas, Appl. Math. Computat. **215**, 212 (2009).
51. R. Sassaman, A. Biswas, Nonlinear Dyn. **61**, 23 (2010).
52. R. Sassaman, A. Biswas, Studies in Mathematical Sciences **1**, 30 (2010).
53. R. Sassaman, A. Heidari, F. Majid, A. Biswas, Dynam. Cont. Discr. Impul. Sys. Series A **17**, 275 (2010).
54. R. Sassaman, A. Biswas, Mathematics in Engineering, Science and Aerospace **2**, 99 (2011).
55. R. Sassaman, A. Heidari, A. Biswas, J. Franklin Institue **347**, 1148 (2010).
56. R. Sassaman, A. Biswas, Commun. Nonlinear Sci. Numer. Simulat. **14**, 3239 (2009).
57. A. Esfahani, Appl. Math. Lett. **24**, 204 (2011).
58. A. Biswas, G. Ebadi, M. Fessak, A.G. Johnpillai, S. Johnson, E.V. Krishnan, A. Yildirim, Iranian Journal of Science and Technology, Transaction-A. **36A**(4), 431(2012).

59. A. Biswas, A. Yildirim, T. Hayat, O.M. Aldossary, R. Sassaman, Proceedings of the Romanian Academy, Series A, **13**, 32 (2012).
60. H. Triki, A. Yildirim, T. Hayat, O.M. Aldossary, A. Biswas, Proceedings of the Romanian Academy, Series A, **13**, 103 (2012).
61. G. Ebadi, A.H. Kara, M.D. Petkovic, A. Yildirim, A. Biswas, Proceedings of the Romanian Academy, Series A, **13**, 215 (2012).
62. A.G. Johnpillai, A. Yildirim, A. Biswas, Romanian Journal of Physics **57**, 545 (2012).
63. H. Triki, S. Crutcher, A. Yildirim, T. Hayat, O.M. Aldossary, A. Biswas, Romanian Reports in Physics **64**, 367 (2012).
64. H. Triki, A. Yildirim, T. Hayat, O.M. Aldossary, A. Biswas, Romanian Journal of Physics **57**, 1029 (2012).
65. L. Girgis, D. Milovic, S. Konar, A. Yildirim, H. Jafari, A. Biswas, Romanian Reports in Physics **64**, 663 (2012).
66. H. Triki, A. Yildirim, T. Hayat, O.M. Aldossary, A. Biswas, Romanian Reports in Physics **64**, 672 (2012).
67. S. Crutcher, A. Oseo, A. Yildirim, A. Biswas, Journal of Optoelectronics and Advanced Materials **14**, 29 (2012).
68. A. Biswas, K. Khan, A. Rahaman, A. Yildirim, T. Hayat, O.M. Aldossary, Journal of Optoelectronics and Advanced Materials **14**, 571 (2012).
69. G. Ebadi, N. Yousefzadeh, H. Triki, A. Yildirim, A. Biswas, Romanian Reports in Physics **64**, 915 (2012).
70. R. Kohl, A. Biswas, D. Milovic, E. Zerrad, Optics and Laser Technology **40**(4), 647 (2008).