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SURFACE WAVES IN AN ISOTROPIC SEMI-INFINITE BODY

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Abstract. We examine the elastic waves excited on the plane surface of an isotropic body (half-space) by an oscillatory force which decreases exponentially with the distance from the surface. We define the "H/V"-ratio as the ratio of the intensity of the waves with polarization parallel to the surface (horizontal waves) to the intensity of the waves with polarization perpendicular to the surface (vertical waves). It is shown that this ratio may exhibit a maximum value or a resonance. Under reasonable assumptions, the resonance frequency may be close to the frequency of the in-plane transverse (shear) waves. It may serve to determine the Poison ratio, or to characterize the sub-surface structure of the body. The resonance is particularly apparent for a vertical force.

Key words: Rayleigh Waves; Surface waves; H/V ratio; Resonances.

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Recently there is a great deal of interest in the method of the "H/V"-ratio for assessing the elastic properties of soils by means of their response to external excitations [1]-[10]. We analyse herein the surface waves excited in an elastic isotropic body with a plane surface (half-space) by an oscillatory force which decreases exponentially with the distance from the surface. The "H/V"-ratio, defined as the ratio of the intensity of the horizontal waves (waves with polarization parallel to the surface) to the intensity of the vertical waves (waves with polarization perpendicular to the surface), may exhibit a resonance peak at a frequency which, under reasonable assumptions, may be close to the frequency of the in-plane transverse (shear) waves.

Usually, the "H/V"-ratio is investigated as the ellipticity of the surface waves (the ratio of the amplitudes moduli) under various boundary conditions. As it is well known, the Rayleigh waves[11] for a homogeneous half-space with a free surface do not exhibit a resonance. The classical theory of Love waves,[12] which assumes a layer super-imposed over a half-space, was recently extended by Malischewsky and Scherbaum[13] to compressible media and various boundary conditions, in order to look for a resonance with a wavelength related to the layer thickness. In the present paper we analyse a different problem, that of the surface waves excited by an external force which decreases exponentially with the distance from the surface. We may call them "forced waves", in contrast with "free waves" which appear as "eigenmodes" of the Navier-Stokes equation with boundary conditions. We find that the "H/V"-ratio of the corresponding waves intensities may exhibit a resonance, which, most likely, may be close to the frequency of the in-plane transverse (shear) waves, as indicated by some experimental works (see, for instance, Ref. [2]). The resonance is particularly apparent for a vertical force.

As it is well known,[14] the equation of elastic waves in an isotropic body (Navier-Stokes equation) is given by

$$\ddot{\mathbf{u}} = c_t^2 \Delta \mathbf{u} + (c_l^2 - c_t^2)\,\mathrm{grad} \cdot \mathrm{div}\,\mathbf{u} + \mathbf{F} \tag{1}$$

where $\mathbf{u}$ is the local displacement, $c_{t,l}$ are the velocities of the transverse and, respectively, longitudinal waves and $\mathbf{F}$ is an external force (per unit mass). Although slightly different from the notations in Theoretical Seismology, we use here the notations of Ref. [14] which serve better our problem and aims. The waves velocities are given by

$$c_t^2 = \frac{E}{2\rho(1+\sigma)} \; , \; c_l^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)} \tag{2}$$

where E is Young's modulus, σ is Poison's ratio ($-1 < \sigma < 1/2$) and ρ is the density of the body.

We consider surface waves in a half-space $z < 0$ excited by an external force

$$\mathbf{F} = -\mathbf{f} e^{-i\Omega t} \delta(\mathbf{r}) e^{\kappa z} \tag{3}$$

where $\mathbf{f}$ is a force per unit mass and unit area, Ω is the frequency of the force, κ is an attenuation coefficient and $\mathbf{r} = (x, y)$ are in-plane coordinates. This would correspond to surface waves excited at Earth's surface by seismic waves or other external perturbations. The localization of the force ($\delta(\mathbf{r})$-term) means that we are interested in surface waves which are far away form the source of excitation. We emphasize that the external force given by equation (3) differs from what is usually called "one of Lamb's problem"[15] by the exponential $e^{\kappa z}$; usually, Lamb's problem[16] assumes a force localized on the surface with respect to the coordinate z too, *i.e.* Lamb's force includes an additional factor $\delta(z)$. Since there could exist doubts about the existence of an external force as the one given by equation (3), with regard to its exponential z-dependence, we give in **Appendix** a sketch of some arguments about the possible existence of such external forces.

We look for solutions of the form $\mathbf{u} \sim e^{i\mathbf{kr}} e^{\kappa z}$ for equation (1) (the only admissible) and introduce the notation $\mathbf{u} = (u_l, u_t, u_v)$ and $\mathbf{k} = (k, 0)$. In addition we assume $\mathbf{f} = (f_l, 0, f_v)$ (*i.e.* a vanishing in-plane transverse component of the force), for simplicity. Equation (1) becomes

$$\ddot{u}_l = (-c_l^2 k^2 + c_t^2 \kappa^2) u_l + i\kappa k (c_l^2 - c_t^2) u_v - f_l e^{-i\Omega t} \ ,$$
$$\ddot{u}_v = (-c_t^2 k^2 + c_l^2 \kappa^2) u_v + i\kappa k (c_l^2 - c_t^2) u_l - f_v e^{-i\Omega t} \ , \tag{4}$$
$$\ddot{u}_t = c_t^2 (-k^2 + \kappa^2) u_t \ .$$

The homogeneous part of equations (4) (f_l, $f_v = 0$) have two distinct eigenfrequencies given by $\omega_{l,t}^2 = c_{l,t}^2 (k^2 - \kappa^2)$, ($k > \kappa$), corresponding to the eigenmodes. It is easy to see that the amplitudes of these eigenmodes go like $u_l \sim ik$, $u_v \sim \kappa$ and, respectively, $u_l \sim \kappa \,, u_v \sim -ik$.

In order to fix notations and further reference points that will be employed subsequently in this paper, we sketch here the standard derivation of the Rayleigh waves (Landau and Lifshitz (2005)). We put $\omega^2 = c_l^2 (k^2 - \kappa_l^2) = c_t^2 (k^2 - \kappa_t^2)$ and take the linear combination

$$u_l = (ikAe^{\kappa_l z} + \kappa_t B e^{\kappa_t z}) e^{ikx} \ ,$$
$$u_v = (\kappa_l A e^{\kappa_l z} - ikBe^{\kappa_t z}) e^{ikx} \tag{5}$$

in order to satisfy the boundary conditions $\sigma_{iz} = 0$ at the free surface $z = 0$, where

$$\sigma_{ij} = \frac{E}{1+\sigma} (u_{ij} + \frac{\sigma}{1-2\sigma} u_{ll} \delta_{ij}) \tag{6}$$

is the stress tensor and $u_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2$ is the strain tensor. These are the well-known Rayleigh waves [11]. From the boundary conditions $\sigma_{iz} = 0$ we get $u_t = 0$ and the equations

$$2i\kappa_l k A + (k^2 + \kappa_t^2) B = 0 \ ,$$
$$\left[\sigma(k^2 + \kappa_l^2) - \kappa_l^2\right] A + i(1-2\sigma)\kappa_t k B = 0 \ . \tag{7}$$

It is easy to see that the ω_l-solution corresponds to $curl\mathbf{u} = 0$ and the ω_t-solution corresponds to $\operatorname{div}\mathbf{u} = 0$. Making use of $c_l^2 (k^2 - \kappa_l^2) = c_t^2 (k^2 - \kappa_t^2) = \omega^2$ and of equations (2) the second equation (7) can also be written as

$$(k^2 + \kappa_t^2) A - 2i\kappa_t k B = 0 \ , \tag{8}$$

so that equations (7) have solutions provided

$$(k^2 + \kappa_t^2)^2 = 4\kappa_l \kappa_t k^2 \ . \tag{9}$$

We introduce the variable ξ defined by $\omega = c_t \xi k$ and, making use of equations (2), we get

$$\kappa_l^2 = (1 - c_t^2 \xi^2 / c_l^2) k^2 = \left[1 - \tfrac{1-2\sigma}{2(1-\sigma)} \xi^2\right] k^2 \ , \ \kappa_t^2 = (1 - \xi^2 \ ) k^2 \ . \tag{10}$$

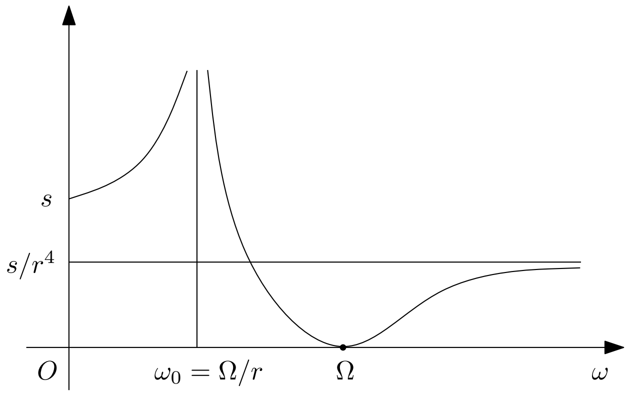

Figure 1 – The resonance of the "H/V"-ratio at frequency $\omega_0 = \Omega/r$.

Now, equation (9) becomes the well-known equation

$$\xi^6 - 8\xi^4 + 8\frac{2-\sigma}{1-\sigma}\xi^2 - \frac{8}{1-\sigma} = 0 \, . \tag{11}$$

This equation has a solution close to unity, $\xi \simeq 1$, for $0 < \sigma < 1/2$. It follows that $\kappa_l \sim k$ and $\kappa_t \sim 0$. The ratio of the two amplitudes is

$$\frac{A}{B} = 2i\frac{\sqrt{1-\xi^2}}{2-\xi^2} \, , \tag{12}$$

so the amplitude of the κ_l-wave (A) is much smaller than the amplitude of the κ_t-wave (B). The main surface wave is a shallow wave with a large penetration depth ($\kappa_t \simeq 0$). This is essentially the Rayleigh's theory of surface waves. As we can see easily from equation (12), the A/B-ratio does not exhibit any resonance (does not depend on frequency, as it is well known).

We pass now to solving equations (4) with the force term (inhomogeneous equations). The solution is of the form $\mathbf{u} \sim e^{i\mathbf{kr}}e^{\kappa z}e^{-i\Omega t}$, where κ is the attenuation coefficient of the force. We get easily

$$u_l = \frac{(\Omega^2 - c_t^2k^2 + c_l^2\kappa^2)f_l - i\kappa k(c_l^2 - c_t^2)f_v}{\Delta} \, ,$$
$$u_v = \frac{(\Omega^2 - c_l^2k^2 + c_t^2\kappa^2)f_v - i\kappa k(c_l^2 - c_t^2)f_l}{\Delta} \tag{13}$$

for the amplitudes of the displacement, which are Fourier components depending on Ω, k and parameter κ, where $\Delta = \left[\Omega^2 - c_l^2(k^2-\kappa^2)\right]\left[\Omega^2 - c_t^2(k^2-\kappa^2)\right]$. We define the "$H/V$"-ratio as $H/V = |u_l|^2/|u_v|^2$. It is convenient to introduce the notation $s = f_l^2/f_v^2$. We get

$$H/V = \frac{(\Omega^2 - c_t^2k^2 + c_l^2\kappa^2)^2 s + \kappa^2k^2(c_l^2 - c_t^2)^2}{(\Omega^2 - c_l^2k^2 + c_t^2\kappa^2)^2 + \kappa^2k^2(c_l^2 - c_t^2)^2 s} \, . \tag{14}$$

The quantity H/V as given by equation (14) has a complex behaviour as a function of frequency Ω, depending on the wavevector k, the attenuation coefficient κ, the wave velocities and the ratio s of the force components. This function may exhibit a more or less peaked maximum, or a resonance. We examine here a few reasonable cases for particular values of these parameters.

It is likely that the attenuation coefficient κ in the expression of the force is very small. We assume here that this is the case. The surface waves given by equation (13) and the "H/V"-ratio given by equation (14) acquire then simple expressions. They are given by

$$u_l \simeq \frac{f_l}{\Omega^2 - c_l^2 k^2}\,, \quad u_v \simeq \frac{f_v}{\Omega^2 - c_t^2 k^2} \tag{15}$$

and

$$H/V \simeq \frac{(\Omega^2 - c_t^2 k^2)^2 s}{\left(\Omega^2 - c_l^2 k^2\right)^2}\,. \tag{16}$$

It is convenient to introduce the frequency $\omega = c_t k$, corresponding to in-plane (shear) waves. Equation (16) can then be written as

$$H/V \simeq \frac{(\Omega^2 - \omega^2)^2 s}{\left(\Omega^2 - r^2\omega^2\right)^2}\,, \tag{17}$$

where we find it convenient to use the ratio $r = c_l/c_t$. A plot of the H/V-ratio *vs.* ω is given in Fig. 1. We can see from equation (17) that the "H/V"-ratio exhibits a resonance for $\omega_0 = \Omega/r = (c_t/c_l)\Omega$. If we take $\Omega = c_l k$, this resonance lies in the vicinity of the in-plane transverse (shear) wave frequency $\omega_0 = c_t k$, in agreement with previous results (see, for instance, Ref. [2]).

For $s = 0$ equation (14) gives

$$H/V = \frac{c_t^2 \kappa^2 \omega^2 (r^2 - 1)^2}{(\Omega^2 - r^2\omega^2)^2}\,, \tag{18}$$

and one can see that the resonance is rather sharp, due to the fact that κ is small. If we take $\Omega = c_t k$ the resonance is placed at $\omega_1 = c_t^2 k/c_l = \omega_0/r$. For $s \to \infty$ the resonance disappears.

The case of a vanishing κ may correspond to $\kappa = \kappa_t = \sqrt{1-\xi^2}k$ for $\xi \simeq 1$. We may also use $\kappa = \kappa_l = \sqrt{1-\xi^2/r^2}k$ for $\xi \simeq 1$, and equation (14) becomes

$$H/V \simeq \frac{\left[\Omega^2 + (r^2-2)\omega^2\right]^2 s + (r^2-1)^3\omega^4/r^2}{\left[\Omega^2 - (r^2+1/r^2-1)\omega^2\right]^2 + s(r^2-1)^3\omega^4/r^2}\,. \tag{19}$$

This expression has a rather broad maximum. For $s = 0$ equation (19) exhibits a resonance at $\omega \simeq (r^2+1/r^2-1)^{-1/2}\Omega = (1+1/r^4-1/r^2)^{-1/2}\omega_0$ which is greater than ω_0 ($r^2 > 2$). For $s \to \infty$ the maximum of (19) disappears.

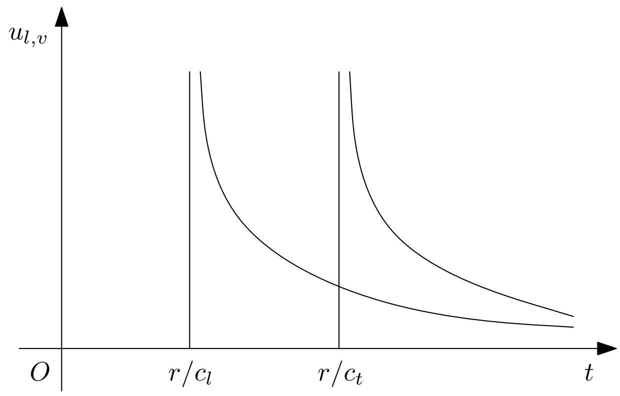

Figure 2 – The wave fronts of the excited surface waves.

A small but finite value of the parameter κ shifts the resonance frequency and smooths out the resonance, giving it a small width. The frequency Ω is not necessary related to the frequency of the elastic waves on the surface, or with the frequency of the in-depth waves of excitations. If the force is a superposition of various frequencies Ω then the resonance is smoothed out and gets a finite width.

It is worth computing the (inverse) Fourier transforms of the displacements given by equation (15). We assume that the external force is smoothly distributed with an average $\overline{f}_{l,v}$ over a large range $\Delta\omega$ of frequencies, *i.e.* $1/\Delta\omega$ is approximately the duration of the external pulse. For vanishing κ we may omit the z-dependence. Then, the displacements as function of position and time are given approximately by

$$u_{l,v}(\mathbf{r},t) \simeq \frac{\overline{f}_{l,v}}{(2\pi)^3\Delta\Omega}\int d\mathbf{k}\int d\Omega\frac{1}{\Omega^2-c_{l,t}^2k^2}e^{-i\Omega t}e^{i\mathbf{kr}}\ . \tag{20}$$

The integration over Ω can be performed immediately, by residues. Similarly, the integration over angles (in $e^{i\mathbf{kr}}$) gives the Bessel function $J_0(kr)$ of the first kind and zeroth order. We get

$$u_{l,t}(\mathbf{r},t) \simeq -\frac{\overline{f}_{l,v}}{4\pi c_{l,t}\Delta\Omega}\int_0^\infty dk\cdot J_0(kr)\sin c_{l,t}kt\ . \tag{21}$$

The remaining integral in equation (21) is a known integral [17]. We get

$$u_{l,t}(\mathbf{r},t) \simeq -\frac{\overline{f}_{l,v}}{4\pi c_{l,t}\Delta\Omega}\frac{\theta(c_{l,t}t-r)}{\sqrt{c_{l,t}^2t^2-r^2}}\ , \tag{22}$$

where $\theta(x)=1$ for $x>1$ and $\theta(x)=0$ for $x<0$ is the step function. This is a cylindrical wave, with an abrupt wavefront at $r=c_{l,t}t$, propagating with the velocities $c_{l,t}$ and decreasing in time as $\sim 1/t$. Its duration is of the order of the pulse duration

$1\Delta\Omega$. An illustration of these waves as functions of time is given in Fig. 2. A finite value of κ smooths out the abrupt wavefront of these waves.

The general solution of equation (1) must include, beside the particular solution given by equations (13) ("forced waves"), the solutions given by equations (5) for the homogeneous equation ("free waves"). The latter are Rayleigh waves with $\Omega^2 = c_l^2(k^2 - \kappa_l^2) = c_t^2(k^2 - \kappa_t^2)$. The amplitudes A and B of these "free waves" are determined by the boundary condition $\sigma_{iz} = 0$ for $z = 0$ (free surface), where the stress tensor includes now the "forced waves" contribution too. This condition leads to a system of two equations, whose determinant is the "Rayleigh" determinant $(k^2 + \kappa_t^2)^2 - 4\kappa_l\kappa_t k^2$ given by equation (9). The resulting solutions for the displacement and the "H/V"-ratio become now much more complicated. However, for Ω close to the transverse wave frequency $c_t\xi k$ the Rayleigh determinant vanishes, and the system of equations for the "free waves" amplitudes A and B is undetermined. The solutions are given in this case essentially by the particular solution corresponding to the "forced waves". In the limit of vanishing κ, the "H/V"-ratio exhibits a rather sharp maximum value which is approximately proportional to $(\xi^2 - r^2)^{-2}$.

We may also comment, finally, upon the usual boundary conditions imposed upon the free surface of a body (half-space). If the body surface is not sharp, then the strain tensor is continuous, and the usual free-stress condition $\sigma_{iz} = 0$ at the surface does not hold any more. Under such circumstances, the solution of the Navier-Stokes equation is the particular solution of the "forced waves" given by equations (13).

In conclusion, we may say that the surface waves excited by an external force in an isotropic half-space may exhibit a resonance of their "H/V"-ratio (horizontal to vertical polarization), which, in most likely conditions, is rather sharp and close to the frequency of the in-plane transverse (shear) waves. The dominant soil frequencies for a response caused by such an external force are the resonance frequency $\omega_0 = \Omega/r$. It may be close to the frequency $c_t k$ of the in-plane (shear) transverse wave (for $\Omega = c_l k$), or to the frequency $c_t^2 k/c_l$ for $\Omega = c_t k$ and vanishing longitudinal force ($f_l = 0$). Similarly, if we assume that Ω is the fundamental frequency of a super-posed layer of thickness d ($\Omega = \pi c_t^0/2d$, where c_t^0 is the transverse wave velocity in the layer), the resonance frequencies is $\pi c_t c_t^0/2c_l d$. These formulae may help in estimating the thickness of the layer or the magnitude of velocities. If we recall that we have denoted here the resonance frequency by $c_t k = 2\pi c_t/\lambda$, we get $\lambda/d = 4c_l/c_t^0$. If the assumption $\lambda/d = 4$ is correct, this would imply $c_l/c_t^0 = 1$. If, on the other side, we assume that the resonance frequency is given by $\Omega/r^0 = c_t^0\Omega/c_l^0 = \pi c_t^0/2d$, then, for, $\Omega = r\omega_0 = 2\pi r c_t/\lambda$, we get $\lambda/d = 4c_l/c_l^0$; for $\lambda/d = 4$ we have $c_l/c_l^0 = 1$. The model presented here can be extended to include damping effects and various other distributions of external forces. The present approach can also be extended to Love's geometry, for a layer super-posed over a half-space. Such questions are left for a

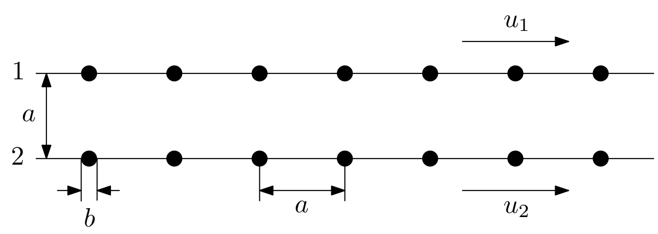

Figure 3 – A schematic representation of two parallel threads of atoms.

forthcoming investigation.

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APPENDIX: AN EXPONENTIALLY DECAYING EXTERNAL FORCE

We consider a semi-infinite isotropic elastic solid (half-space) extending over the region $z < 0$. We are interested in its elastic motion (elastic waves) under the action of an external force, in particular an external damping force of the form $f \sim e^{\kappa z}$, where κ is an attenuation (damping) coefficient. The reasonability of such an external force is sometimes questioned. We sketch here an argument that such a force is plausible within the general definition of an elastic solid.

First, we resort to the general atomic structure of solid matter. In a simplified model we may focus attention upon two parallel threads of atoms, arranged at their equilibrium positions in a generic solid, as shown in Fig. 3. We recall that one of the basic assumptions of the linear elasticity consists of admitting only local displacements which, though they may be large, must have a small spatial derivative, *i.e.* they must vary very slowly within the solid. This means that atoms in thread 1 in Fig. 3 may move only within the inter-atomic spaces of thread 2 (each such space of length a in Fig. 3). If we look to a large stack of such many parallel threads, it is easy to see indeed that the displacements may get high values, but certainly they vary slowly over large distances. This amounts to disregard the friction, or other losses in an ideal elastic solid.

For external forces the situation may be different, especially near their location, where we may expect abrupt changes in atomic positions. Suppose that such a force f acts along the thread 1 and causes a displacement u_1 of this thread. Atoms in thread 1 may pass over atoms in thread 2. As long as atoms in thread 1 move within the inter-atomic space of thread 2 we may expect little displacement of thread 2. Each time, however, that atoms in thread 1 pass over atoms in thread 2 we may expect a certain contribution to the displacement u_2 of thread 2. It is reasonable to assume that such a displacement is of the order of the atomic size denoted by b in Fig. 3. Suppose that there are n such pass-over steps. We get approximately a displacement

$$u_2 = nb \, . \tag{23}$$

Now, the number n is obviously proportional to u_1/a, so we have

$$u_2 = \frac{b}{a} u_1 \, . \tag{24}$$

Suppose now that we take a stack of parallel chains, the first, denoted by u_0 located at $z = 0$ and the n-th, denoted by u_n located at z. Obviously, we have

$$u_n = \left(\frac{b}{a}\right)^n u_0 \tag{25}$$

by iterating equation (24), or

$$u(z) = u_0 e^{-\frac{z}{a}\ln(b/a)} = u_0 e^{\kappa z} \, , \tag{26}$$

where $\kappa = -\frac{1}{a}\ln(b/a)$; we have taken into account that $z < 0$, $\ln(b/a) < 0$ and have assumed the same distance a between the threads. Such a damping displacement, caused by an external force, may generate an elastic force which, obviously, has the same damping character: $f \sim e^{\kappa z}$ (through $f_i = \partial\sigma_{ij}/\partial x_j$, where σ_{ij} is the stress tensor). This is, of course the external force. It may even have a abrupt decrease (though not necessarily), for large κ, but, of course, this does not impede on the basic assumption of the elasticity, because its magnitude may be sufficiently small as to preserve the small variations of the elastic displacement.

REFERENCES

1. M. Nogoshi and T. Igarashi, J. Seism. Soc. Japan **23**, 264-280 (1970).
2. M. Nogoshi and T. Igarashi, J. Seism. Soc. Japan **24**, 26-40 (1971).
3. F.E. Udwadia and M.D. Trifunac, Bull. Seism. Soc. Am. **63**, 1227-1253 (1973).
4. Y. Nakamura, Q. Rep of Railway Technical Research Institute Jpn. **30**, 25-33 (1989).
5. E. Field and K. Jacob, Geophys. Res. Lett. **20**, 2925-2928 (1993).
6. E. Field and K. Jacob, Bull. Seism. Soc. Am. **85**, 1127-1143 (1995).
7. H. Yamanaka, M. Dravinski and H. Kagami, Bull. Seism. Soc. Am. **83**, 1595-1609 (1993).
8. J. Lermo and F.J. Chavez-Garcia, Bull. Seism. Soc. Am. **83**, 1574-1594 (1993).
9. J. Lermo and F.J. Chavez-Garcia, Bull. Seism. Soc. Am. **84**, 1350-1364 (1994).
10. C. Lachette and P.-Y. Bard, J. Phys. Earth **42**, 377-397 (1994).
11. Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17**, 4-11, *Scientific Papers* (1900), vol. **2**, 441-447, Cambridge, London (1885).
12. A.E.H. Love, *Some Problems of Geodynamics* (Cambridge University Press, London, 1926).
13. P.G. Malischewsky and F. Scherbaum, Wave Motion **40**, 57-67 (2004).
14. L. Landau and E. Lifshitz, *Theory of Elasticity* (Elsevier, NY, 2005).

15. M.A. Pelissier, H. Hoeber, N. van de Coevering and I.F. Jones, eds., *Classics of Elastic Waves Theory*, Society of Exploration Geophysicists (Tulsa, OK, 2007).
16. H. Lamb, Phil. Trans. Roy. Soc. (London) **A203**, 1-42 (1904).
17. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, NY, p. 709, 6.671, 2000).