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INVERSES OF LANGEVIN, BRILLOUIN AND RELATED FUNCTIONS: A STATUS REPORT

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Abstract. This paper gives a coherent and comprehensive review of the results concerning the inverses of Langevin $L(x)$ and of Brillouin functions $B_J(x)$ and the inverses of $L(x)/x$ and $B_J(x)/x$, with main focus on the most recent developments. As these functions are used in several fields of physics, without evident interconnections - magnetism (ferromagnetism, nanomagnetism, hysteretic physics), rubber elasticity, rheology, solar energy conversion - the new results are not always efficiently transferred from a domain to another. The increasing accuracy of experimental investigations claims an increasing accuracy in the knowledge of these functions, so it is important to compare the accuracy of various approximants and even to obtain, in some cases, the exact form of the inverses of $L(x)$, $B_J(x)$, $L(x)/x$ and $B_J(x)/x$, using the recently created theory of generalized Lambert functions. The paper contains also some new results, concerning both exact and approximate forms of the aforementioned inverse functions.

Key words: Generalized Lambert functions, Weiss equation, magnetization, rubber elasticity, ferrofluids, analytical approximations.

1. INTRODUCTION

In recent years, the progress registered in two different fields - (1) the increasing accuracy of experimental investigation in several domains of physics, like rubber elasticity, ferrofluids or magnetism and (2) the possibility of solving a class of transcendental equations, using the theory of newly introduced generalized Lambert functions - stimulated the interest in the study of inverses of Langevin, Brillouin and related functions. Besides exact results, a number of approximants, of various precisions or degrees of sophistication, exist, with important applications in rheology, magnetism and other fields. A comprehensive comparative study of the approximants, of their accuracy and applicability, and of their connections with the exact results - when they are available - is missing, and the main goal of this paper (which is actually a shortened version of [1]) it to fill this gap. Older results are discussed in the light of new developments. Some original results are also included. Our intention was to give a coherent presentation of the status of the art in this domain, of interest for experimentalists, theorists and mathematicians.

The structure of this paper is the following. In Section 2, the physical relevance of Brillouin and Langevin functions is discussed, and the importance of their inverses is illustrated, by two simple examples. The applications of direct and inverse Langevin and Brillouin functions, which are rarely analyzed together, are given in the next section. The basic properties of direct and inverse Langevin and Brillouin functions - their asymptotic behavior, or for small values of arguments, the algebraic form of Brillouin functions, as well as the ”higher-order Langevin functions - are presented in Section 4. Exact results concerning L^{-1}, B_J^{-1} and related functions, like the inverse of B_J/x, are exposed in Section 5. In the next one, the approximates of B_J^{-1} proposed by several authors are critically discussed, and a recent accurate formula is presented. The last section is devoted to conclusions.

2. PHYSICAL RELEVANCE OF BRILLOUIN AND LANGEVIN FUNCTIONS

The Langevin and Brillouin functions, defined as:

$$L(x) = \coth x - \frac{1}{x} \tag{1}$$

and respectively

$$B_J(x) = \frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}x\right) - \frac{1}{2J}\coth\left(\frac{1}{2J}x\right),\ J = \frac{2n+1}{2},\ n = 0,\ 1,\ 2,... \tag{2}$$

with x real, where introduced in science in the context of paramagnetism and ferromagnetism. Almost any textbook in these domains contains the description of the following two simple results (we adopted here the approach and notations of [2]):

(1) For a classical magnetic moment $\overrightarrow{\mu}$ in an external field $\overrightarrow{B}$, the Zeeman energy is:

$$U = -\overrightarrow{\mu}\overrightarrow{B} = -\mu B\cos\theta \tag{3}$$

with θ- the angle between $\overrightarrow{\mu}$ and $\overrightarrow{B}$ (oriented along the OZ axis). The magnetization at thermal equilibrium is proportional to the thermal average of $\cos\theta$:

$$\langle\cos\theta\rangle = \frac{\int e^{-\beta U}\cos\theta d\Omega}{\int e^{-\beta U}d\Omega} = L(X) \tag{4}$$

with X and β defined as:

$$X = \frac{\mu B}{k_B T},\ \beta = \frac{1}{k_B T}. \tag{5}$$

(2) For a perfect paramagnet - a system of N non-interacting quantum magnetic moments $\overrightarrow{\mu}$ with maximum spin J in an external magnetic field $\overrightarrow{B}$ at temperature T

- the magnetization M, obtained as a quantum thermal average of $N\mu_z$, is:

$$\frac{M}{M_{\max}} \equiv m = B_J(x) \tag{6}$$

with x defined as:

$$x = \frac{\overline{\mu}BJ}{k_B T},\ \overline{\mu} = g_J \mu_B, \tag{7}$$

g_J being the Landé factor, and μ_B the Bohr magneton.

With some caution requested by the fact that the average in (4), respectively (6) is taken in classical, respectively quantum context, it is clear that both $L(x)$ and $B_J(x)$ characterize the alignment of a magnetic moment in an external field, so they behave as order parameters. (As we shall see in Section 3, in some models of ferrofluids, the higher-order Langevin functions, defined later on in eq. (37), are also connected to orientational order parameters.) Indeed, both $L(x)$ and $B_J(x)$, for $x > 0$, being thermal averages of $\langle \cos\theta \rangle$, $0 < \theta < \pi/2$, have quite similar shapes, as functions of x : they are monotonically increasing functions, starting from zero, and reaching asymptotically the value 1. As $L(x)$ and $B_J(x)$ describe, in the simple magnetic models just mentioned, the same physics - in classical and, respectively, in quantum frame - $L(x)$ is the classical limit of $B_J(x)$, corresponding to "large spins":

$$L(x) = \lim_{J\to\infty} B_J(x) \tag{8}$$

For a system of interacting quantum magnetic moments, in the mean-field approximation - the Weiss model - the magnetization is given by a formula somewhat similar to eq. (6), but more complex:

$$m(t,h) = B_J\left(\frac{m(t,h)+h}{t}\right) \tag{9}$$

(see [2], eq. (6.14)), where the reduced quantities are:

$$t = \frac{T}{T_c},\ h = \frac{\overline{\mu}H}{2k_B T_c},\ m = \frac{M}{M_0}, \tag{10}$$

with $M_0 = M_{\max} = N\overline{\mu}S$, λ - the Curie constant and T_c - the critical temperature:

$$T_c = \lambda \frac{N\overline{\mu}^2}{4k_B},\ M_0 = N\overline{\mu}J \tag{11}$$

Putting, in (9),

$$m(t,0) = m(t) \tag{12}$$

the equation of state becomes, for $h = 0$:

$$m(t) = B_J\left(\frac{m(t)}{t}\right) \tag{13}$$

Now, we can illustrate the usefulness of the inverses of the functions $B_J(x)$ and $B_J(x)/x$ in the Weiss theory of magnetism. Taking B_J^{-1} in both sides of eq. (13), we get:

$$\frac{B_J^{-1}(m)}{m} = \frac{1}{t} \tag{14}$$

so, we can separate the thermodynamic variables m and t in the equation of state (see also [3], eq. (12)). However, we cannot obtain, in this way, the explicit form of the function $m(t)$.

If we put:

$$\frac{m(t)}{t} = \zeta(t) \tag{15}$$

eq. (13) can be written as:

$$t = \frac{B_J(\zeta)}{\zeta} \tag{16}$$

Denoting by $\beta_J(x)$ the inverse of the function $B_J(x)/x$, we get from (15), (16):

$$m(t) = t\beta_J(t) \tag{17}$$

which gives explicitly the dependence of magnetization on temperature - a relation very useful for experimentalists. Clearly, the previous remarks remain valid in the classical limit, $J \to \infty$ (see (8)).

This is a simple example illustrating the usefulness of the inverses of functions $B_J(x)$, $L(x)$, $B_J(x)/x$, $L(x)/x$ in magnetism. It is interesting to note that, for some subtle reasons, discussed by Callen and Shtrikman [4], the Weiss theory gives quantitatively accurate predictions, "far beyond any reasonable expectations" [4], so the applications in magnetism of the aforementioned functions go far beyond the Weiss model. More than this, they are not limited to magnetism, as we shall see.

We shall finish this section with a terminological remark. It is interesting to mention that the "Brillouin functions" were actually introduced in physics by Debye and by other authors, in the context of old quantum theory, see [5, 6]. Brillouin used these functions in the frame of quantum mechanics. As the term "Debye function" already existed, in the theory of specific heat of solids, the new functions took Brillouin's name. However, some prominent physicists, like Wannier, call both $L(x)$ and $B_J(x)$ "Langevin functions" [7].

3. APPLICATIONS OF DIRECT AND INVERSE LANGEVIN AND BRILLOUIN FUNCTIONS

Although the Langevin function was introduced in physics in the context of magnetism, the direct and inverse Langevin functions are used in a multitude of domains, sometimes far away from their initial field of application: polymer science, ferrofluids, solar energy conversion.

In order to understand the physical basis of the presence of Langevin function, as an essential mathematical tool, in two so different domains, like rubber elasticity and paramagnetism, the analysis of an idealized model of rubber-like chain, given in Kubo's treatise of statistical mechanics [8], is very illustrative.

Let us start the applications of the Langevin function in rheology (polymers, magnetic fluids) with two examples from rubber elasticity. With the development of very precise experimental techniques in the last 30 years, like single-molecule force spectroscopy (SMFS), the elastic strain energy u_c and the force (f_c) - displacement $\left(\frac{r}{L_0}\right)$ relationship for a chain, with L_0 - the contour length of the chain and N - the number of rigid kinks in the chain, which can be written in terms of L^{-1} [9, 10]:

$$u_c = kTN\left(xL^{-1}(x) + \ln\frac{L^{-1}(x)}{\sinh L^{-1}(x)}\right), \quad x = \frac{r}{L_0} \tag{18}$$

$$f_c = \frac{kTN}{L_0}L^{-1}\left(\frac{r}{R}\right) \tag{19}$$

could be measured very precisely; to keep pace with SMFS, the theory had to produce more and more precise expressions for the inverse Langevin function L^{-1} [11–13].

We shall refer now to magnetic fluids - materials of great interest for numerous applications in physics, engineering, biology and biotechnology. Both rheological ferrofluids (containing colloidal particles with a permanent ferromagnetic core) and magnetorheological fluids (containing paramagnetic particles) are composed, in general, by single-domain magnetic particles. Consequently, their magnetic behavior is simpler, compared with bulk magnets. In their mathematical description, the Langevin function and "the higher order Langevin functions" L_j (to be defined in eq. (37) of Section 4) play a central role. In the magnetism of solid state physics, with much more complex structures, the central role belongs to Brillouin functions.

Some specific examples of applications of Langevin function in rheology: the distribution function moments of rigid dipoles in a magnetic field at thermodynamic equilibrium, and other similar thermal averages, are written in terms of $L_j(\xi)$ and $L_j(\xi)/\xi$ [14]. The differential magnetic susceptibility of a ferrofluid measured along the external field is ([14, 15]):

$$\chi(\xi) = 3\chi_0\frac{dL(\xi)}{d\xi} \tag{20}$$

where ξ is the Langevin parameter,

$$\xi = \frac{\mu_0\mu H}{kT} \tag{21}$$

In an extension of the kinetic model [16], the one-particle equilibrium distribution giving the orientation of individual colloidal particle, eq. (16) of the aforementioned paper, can be written in terms of L_j. The equilibrium order parameters S_j^{eq} are proportional, in the lowest order, to L_j functions, see Eqns. (19), (20) of [16]. For a generalized Maxwell mesoscopic model of ferrofluids [17], the thermal average of the versor which gives the orientation of the nanoparticle $\langle \overrightarrow{u} \rangle$, eq. (43), and the expression of arbitrary moments of $\langle \overrightarrow{u} \rangle$, as well as the rotational viscosity, eq. (45), are written using the L_j functions.

In a Fokker-Planck equation approach for ferrofluids composed of ferromagnetic ellipsoids, the pressure tensor and viscosities can be expressed in terms of order parameters, which, in some particular cases, can be replaced by L_j functions ([18], Sect. 3.2) In a mean-field kinetic theory of moderately concentrated ferrofluids, the effective diffusion coefficients are written in terms of L_2 [19, 52]. The correctness of these relations was tested by equilibrium and non-equilibrium computer simulations [20]. In various mean-field theory of ferrofluids, (1) the non-equilibrium magnetization can be expressed as a simple function of L_1 and L_2 [21] or (2) the effective magnetic field acting on each chain can be written using the Langevin function, see eq. (11) of [22].

Also, in a chain model, the hydrodynamic equations are expressed in terms of $L(x)/x$ (see [23], Eqns. (10) - (12)), and in a non-interacting model, the modified rotational viscosity is written in terms of Langevin function, eq. (9) in [24], etc.

Last but not least, the Langevin function is of course important in superparamagnetism [25] and nanomagnetism - for instance in the theory of tunnelling magnetoresistance in granular manganites, as the core of nanoparticles is superparamagnetic, see [26], especially eqs. (1-5). The field-induced birefringence of the ferrofluid, in a superparamagnetic (Langevin) model, eq. (3) of [49] is:

$$\Delta n = (\Delta n)_0 \left(1 - 3\frac{L(\xi)}{\xi}\right) \tag{22}$$

Let us move on to a completely different domain - the solar heating, where several problems can be solved using the inverse Langevin function, as noticed Keady [27]. Namely: the fractional time distribution for the clearness K can be written as:

$$F(K) = \frac{\exp(\gamma K) - \exp(\gamma K_{\min})}{\exp(\gamma K_{\max}) - \exp(\gamma K_{\min})}, \; K_{\min} < K < K_{\max} \tag{23}$$

with $K_{\min}$, $K_{\max}$ and $\overline{K}$ - the average value of K - are measurable quantities, but γ, a parameter important for solar heating technology, is not. However, it can be determined solving the equation:

$$\overline{K} = \int_{K_{\min}}^{K_{\max}} \frac{dF}{dK} K dK = \delta \coth \gamma\delta + \sigma - \frac{1}{\gamma} \tag{24}$$

where we put:

$$\sigma = \frac{1}{2}\left(K_{\max} + K_{\min}\right), \ \delta = \frac{1}{2}\left(K_{\max} - K_{\min}\right) \tag{25}$$

Consequently, the solution of (24) can be written as:

$$\gamma = \frac{1}{\delta} L^{-1}\left(\frac{\overline{K} - \sigma}{\delta}\right) \tag{26}$$

Eq. (23) defines a doubly-truncated exponential distribution, which describes also the distribution of earthquakes, of forest-fire sizes, raindrop sizes, reliability modelling, see ref. [1], [3], [6] in [27], so the inverse Langevin function is useful in all these domains, as noticed Keady [27].

Let us discuss now the applications of Brillouin functions. As mentioned in Section 2, they give the equation of state in the Weiss model; they give also the magnetic contribution to heat capacity of such a ferromagnet (see eqs. (22), (23) in [28]). However, the relationship between thermal and magnetic energy per particle, in a Weiss model, is given by the inverse Brillouin function (see eq. (63) and Fig.5 in [29]). Similarly, the Brillouin functions enters in the equation of state of other spin systems: an arbitrary infinite-range spin Hamiltonian [30–32] or an arbitrary anisotropic ferromagnetic spin Hamiltonians [33]; it gives also the sublayer magnetization in multilayer compounds, eq. (4) in [34].

In an interesting approach of defining the Curie temperature, Harrison proposes a positive-feedback model; the hysteretic field, central to this model, is defined through an inverse Brillouin function, eq. (10) of [35]. Applying Harrison's model to the study of phase transitions in magnetocaloric materials [53], the authors limit themselves to the $J = 1/2$ case, in order to avoid numerical methods. With the new analytical approximations for inverse Brillouin functions (see Subsection 6.2), such a restriction can be relaxed.

It is well known that the molecular field theory is frequently used in the study of various magnetic systems to express the higher moments $M_n = \langle (S_z)^n \rangle$ in terms of the first moment $M_1 = \langle S_z \rangle$, the latter quantity being known from empirical magnetization data. However, as Callen and Shtrikman demonstrated, such relations follow only from a particular feature of molecular field theory, which has much wider generality and validity than this theory [36]. In [36], it is also shown that the thermal average $\langle \exp(aS^z) \rangle$ can be written in terms of $\langle S^z \rangle$, if an analytical expression for B_J^{-1} is known. These results were applied by Callen and Callen [37] to magnetostriction, forced magnetostriction and anomalous thermal expansion in ferromagnets. The calculation of temperature dependence of the anisotropy coefficients κ_2, κ_4 was studied by Millev and Fähnle [38–40], at a time when only the inverses of Brillouin function for small values of J $(1/2,\ 1,\ 3/2)$ were known analytically. "Unluckily, these values of J support the second moment M_2 only and, hence, only κ_2 can be found explicitly" (see [40], the comments below eq. (8)). The Callen and Shtrikman program is formally accomplished in these papers, see mainly eqs. (6), (7), (12), (13), (14) in [39]. With explicit expressions of inverse Brillouin functions (see Subsection 6.2), it can be now explicitly accomplished, in an approximate analytical form.

In a large variety of magnetic materials, from carbonaceous solids [41] to dilute magnetic semiconductors [42], the magnetization is well described by a Brillouin function, with a slightly shifted argument ($T \to T + T_0$, with T_0 - an empirical parameter).

At a more sophisticate level, the direct or inverse Brillouin functions are used in theory of helical spin ordering (see [43]), of initial condition of isolated spins in the Ising model [44], [45] or in a recent derivation of scaling laws for the Kondo model [46].

4. BASIC PROPERTIES OF DIRECT AND INVERSE LANGEVIN AND BRILLOUIN FUNCTIONS

As previously mentioned, $L(x)$ and $B_J(x)$ have similar shapes, which can be easily visualized, as:

$$B_{1/2}(x) = \tanh x \tag{27}$$

It is clear from Fig. 1 (and easy to prove mathematically) that:

$$B_{1/2}(x) > B_1(x) > ... > L(x) \tag{28}$$

These inequalities reflect the fact that a smaller magnetic moment can be easier aligned by a certain external field, at a certain temperature, than a larger one, the limiting cases being the smallest ($J = 1/2$) and the largest ($J = \infty$) spin. However, the Brillouin and Langevin functions are qualitatively different: for instance, they can be inverted by solving an algebraic equation (Brillouin) or a transcendental one (Langevin), and the integral of the inverse functions on their domain of definition is convergent, respectively divergent. If we want to speculate, we could comment that the quantum approach is simpler than the classical one.

As the functions are odd:

$$L(-x) = -L(x), \;\; B_J(-x) = -B_J(x) \tag{29}$$

and

$$0 \leqslant L(x), \; B_J(x) < 1, \;\; x \geqslant 0 \tag{30}$$

all the information concerning these functions is contained in the first quadrant.

Near the origin, the Langevin function is:

$$L(x) = \frac{x}{3} + \mathcal{O}\left(x^3\right), \tag{31}$$

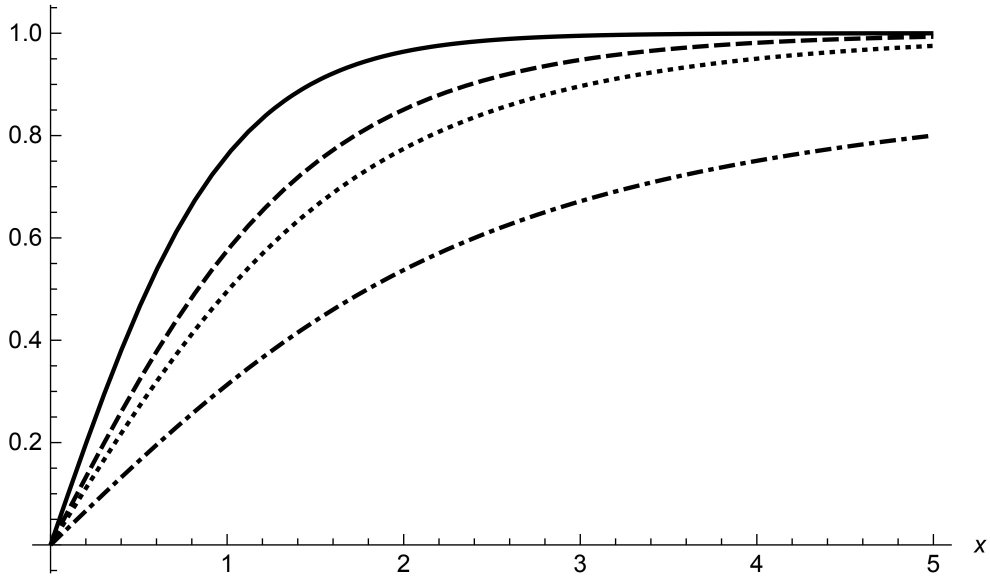

Fig. 1 – The plot of Brillouin functions of indices 1/2 (solid), 1 (dash), 3/2 (dots) and of Langevin function (dot-dash)

a formula obtained from a Taylor series expansion. However, for small values of x, the Lambert continued fraction provides a more accurate analytical approximation for $L(x)$ than the Taylor series, see [47].

For the inverse Langevin function: near the origin,

$$L^{-1}(x) = 3x + \mathcal{O}\left(x^3\right) \tag{32}$$

and asymptotically [48]:

$$L^{-1}(x \to 1) \to \frac{1}{1-x} \tag{33}$$

Other useful relations are:

$$L(x) = \coth x - \frac{1}{x} = \frac{d}{dx} \ln \frac{\sinh x}{x} \tag{34}$$

and:

$$\frac{dL}{dx} = 1 - L^2 - \frac{2L}{x} \tag{35}$$

The inverse Langevin function L^{-1} can be obtained by solving a transcendental equation, (34); actually, as we shall see, L^{-1} is a generalized Lambert function; its expression will be given in the next section.

As Callen et Callen noticed [37], the Langevin function can be written as:

$$L(x) = \frac{I_{\frac{3}{2}}(x)}{I_{\frac{1}{2}}(x)} \equiv L_1(x) \tag{36}$$

(where $I_{\frac{3}{2}}(x)$, $I_{\frac{3}{2}}(x)$ are modified Bessel functions of the first kind) and re-defined as $L_1(x)$. The previous definition of $L_1(x)$ can be extended:

$$L_j(x) = \frac{I_{j+1/2}(x)}{I_{1/2}(x)} \tag{37}$$

and $L_j(x)$ can be considered "higher-order Langevin functions" [37]. The recurrence relation between the modified Bessel functions:

$$I_\nu(\xi) = I_{\nu-2}(\xi) - \frac{2(\nu-1)}{\xi} I_{\nu-1}(\xi) \tag{38}$$

generates a recurrence relation for the L_j functions [14, 50]:

$$L_{j+1}(\xi) = L_{j-1}(\xi) - \frac{2j+1}{\xi} L_j(\xi) \tag{39}$$

Another related function is $L(x)/x$, sometimes noted $L^*(x)$ [49]; as we could see in the previous examples, it appears quite frequently in theory.

Let us examine now the Brillouin functions: near the origin

$$B_J(x) = \frac{J+1}{3J} x + +\mathcal{O}\left(x^3\right) \tag{40}$$

and asymptotically [48]:

$$B_J(x) = 1 - \frac{1}{J\left(1+e^{x/J}\right)}, \quad x \to \infty \tag{41}$$

A useful equation related to $B_J(x)$ is:

$$B_J(x) = \frac{d}{dx} \ln \sum_{s=-J}^{J} \exp\left(-\frac{sx}{J}\right) = \frac{d}{dx} \ln S_J(x) \tag{42}$$

or, equivalently:

$$\int_0^x B_J(y)\, dy = \ln S_J(x) \tag{43}$$

where the function $S_J(x)$ has the form:

$$S_J(x) = \sum_{s=-J}^{-J+1,\ldots J-1,J} \exp\left(-\frac{sx}{J}\right) \tag{44}$$

So, the integral of the Brillouin function is simpler than the function itself. Consequently, $S_J^{-1}(x)$ can be obtained easier than B_J^{-1}, and, if we really obtain it, and if we can write (42) in terms of inverse functions, this could provide us an alternative way of calculating B_J^{-1}. This approach has not been explored yet, although several interesting relations between the integrals of direct and inverse functions were given in literature [9, 48].

Contrary to the previous (Langevin) case, the inverse Brillouin functions can be obtained algebraically. Indeed, with a change of variable:

$$z = e^{\frac{x}{J}} \tag{45}$$

or:

$$x = J \ln z \tag{46}$$

both $B_J(x)$ and $S_J(x)$ become algebraic functions of z. Let us put:

$$B_J(x) = B_J(J \ln z) = \\ = \frac{1}{2J} \frac{(1+2J)\left(z^{1+2J}+1\right)(z-1) - \left(z^{1+2J}-1\right)(z+1)}{(z^{1+2J}-1)(z-1)} = \overline{B}_J(z) = t \tag{47}$$

$$S_J(x) = S_J(J \ln z) = z^J + z^{J-1} + ... + z^{-(J-1)} + z^{-J} = \overline{S}_J(z) = t \tag{48}$$

The roots $z_J(t)$ of the algebraic equation

$$\overline{B}_J(z_J(t)) = t \tag{49}$$

satisfy also the identity:

$$B_J(J \ln z_J(t)) = t \tag{50}$$

Applying in both sides of (50) the inverse of B_J, we get:

$$J \ln z_J(t) = B_J^{-1}(t) \tag{51}$$

So, according to (51), the quantity $x_J(t)$, defined as

$$x_J(t) = J \ln z_J(t) \tag{52}$$

where $z_J(t)$ is a convenient root of (47), is the inverse of B_J :

$$B_J^{-1}(t) = x_J(t) = J \ln z_J(t) \tag{53}$$

The term ”convenient” designates the unique real root of (47), as we shall explain in Subsection 5.2, see the comments subsequent to eq. (82).

Similarly, the convenient root $\widetilde{z}_J(t)$ of the algebraic equation

$$\overline{S}_J(\widetilde{z}_J(t)) = t \tag{54}$$

gives the inverse of S_J :

$$\widetilde{x}_J(t) = J \ln \widetilde{z}_J(t) = S_J^{-1}(t) \tag{55}$$

So, obtaining $z_J(t)$ is equivalent to obtaining $B_J^{-1}(t)$, and obtaining $\widetilde{z}_J(t)$ is equivalent to obtaining $S_J^{-1}(t)$.

Even if the algebraic form of Brillouin functions is the key for obtaining their inverses, it is worth mentioning Katriel's approach of writing $B_J^{-1}(t)$ as a continued-fraction [51].

Near the origin, $B_J^{-1}(t)$ behaves like:

$$B_J^{-1}(x) = \frac{3J}{J+1}x + +\mathcal{O}\left(x^3\right) \tag{56}$$

and asymptotically:

$$B_J^{-1}(t \to 1) \to J \ln \frac{1}{1-t} \tag{57}$$

a relation given by Kroger [48] in a slightly different form, and re-obtained in Subsection 5.2, using another approach.

5. EXACT RESULTS CONCERNING THE INVERSION OF $B_J(x)$, $L(x)$, $B_J(x)/x$, $L(x)/x$

5.1. GENERALIZED LAMBERT FUNCTIONS APPROACH

The generalized Lambert functions, recently introduced by Mezö, Baricz and Mugnaini [54, 55], provide the basis for obtaining explicit solutions (series expansions) for several classes of transcendental equations of mathematical physics. Some of their physical applications are given in [56, 57]. In this subsection, we shall define these functions and shortly present their relevance for magnetism and rheology.

If the Lambert function $W(a)$ [58] is the solution of the transcendental equation:

$$xe^x = a \ \to x = W(a), \tag{58}$$

the generalized Lambert function $x = W(t_1, t_2, ...t_n; s_1, s_2, ...s_m; a)$ is the solution of a similar, but more complicated equation:

$$e^x \frac{(x-t_1)...(x-t_n)}{(x-s_1)...(x-s_m)} = a \tag{59}$$

where the parameters $t_1, ..., t_n$ ("upper parameters") and $s_1..., s_m$ ("lower parameters") are supposed to be real [54, 55]. A formula obtained in [54, 55], useful in the theory of Langevin and Brillouin functions, is:

$$W(\tau;\sigma;a) = \tau - (\tau - \sigma)\sum_{n=1}^{\infty} \frac{L_n'(n(\tau-\sigma))}{n} e^{-n\tau} a^n \tag{60}$$

where $\sigma \neq \tau$, and L_n' is the first derivative of the $n-$th order Laguerre polynomial. Eq. (60) can be written as:

$$W(\tau;\sigma;a) = \tau + W_{-a\exp(-\tau)}\left(a(\tau-\sigma)e^{-\tau}\right) \tag{61}$$

where $W_r(x)$, named the $r-$ Lambert function, is the inverse of the function $xe^x + rx$ (r is a real number), so

$$xe^x + rx = a \ \to x = W_r(a) \tag{62}$$

Among the branches of W_r, a special role is played by $W_{1/e^2}(x)$ [54]. For $x = -4/e^2$, $W_{1/e^2}(x)$ has a unique property: it is continuos (as in any other point on the real line) but is not differentiable, and

$$W_{1/e^2}\left(-4/e^2\right) = -2 \tag{63}$$

As we shall see later, it corresponds to the spontaneous magnetization of a Weiss ferromagnet [59].

5.2. THE INVERSE OF $L(x)$ AND $L(x)/x$

The inverse Langevin function is a solution of eq. (59), with one upper and one lower parameter. More exactly: if $L(x) = a$, the function $L^{-1}(a)$ is:

$$L^{-1}(a) = -2W\left(\frac{2}{a+1};\frac{2}{a-1};\frac{a-1}{a+1}\right) \tag{64}$$

Also, it is a simple exercise to show that the inverse of the function $L(x)/x$, *i.e.* the solution in x of the equation $L(x) = \alpha x$, is a generalized Lambert function with two upper and two lower parameters - a case when a general formula is not available. Its explicit form was obtained long time ago by Siewert and Burniston [60], but their result is quite inconvenient for physical applications.

5.3. INVERSION OF $B_J(x)/x$

Concerning the function $\frac{B_J(x)}{x}$, the exact expression of its inverse was obtained only for $J = 1/2$, *i.e.* for the function $\frac{\tanh x}{x}$ [59, 61]. As already explained, the Weiss equation for $J = 1/2$ (eq. (22) of [61] or eq. (10) of [59]) takes the form:

$$\frac{\tanh\zeta(t)}{\zeta(t)} = t \tag{65}$$

So, to invert the function $(\tanh\zeta)/\zeta$ means to obtain the function $\zeta(t)$ and, consequently, the magnetization $m(t)$. Finally, $m(t)$ can be written as:

$$m(t) = t\zeta(t) = \frac{t}{2}W\left(\frac{2}{t};-\frac{2}{t};-1\right) \tag{66}$$

According to (60), (61):

$$m(t) = 1-2\sum_{n=1}^{\infty}\frac{L_n'(4n/t)}{n}\left(-e^{-2/t}\right)^n = \frac{2}{t}+W_{\exp(-2/t)}\left(-\frac{4}{t}\exp(-2/t)\right) \tag{67}$$

It is easy to see from eq. (60) that the condition

$$m(0) = 1 \tag{68}$$

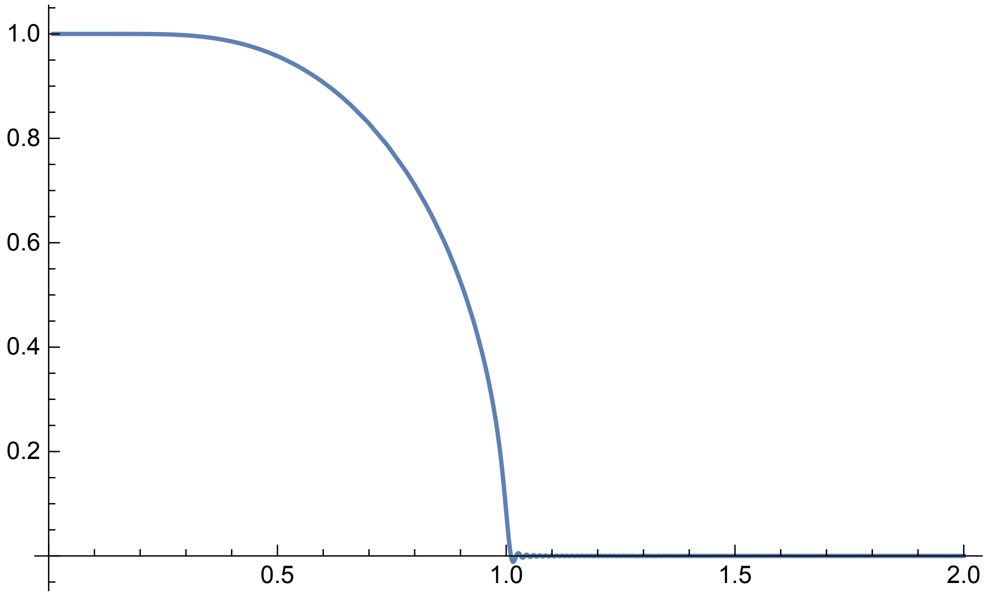

Fig. 2 – The spontaneous magnetization $m(t)$, according to eq. (68), when the first $n = 1000$ terms of the series are took into account. If $n > 2000$, the oscillations for $t > 1$ cannot be perceived by naked eye.

is fulfilled, as $L_1'(x) = -1$ and $\lim_{t \to 0} e^{-2/t} = 0$. Due to the exponential term, the series in the r.h.s. of (60) is rapidly convergent.

If the temperature takes its critical value, $t = 1$, the index of W_r takes also its critical value (see eq. (63)), namely:

$$r = \frac{1}{e^2} \tag{69}$$

In this case, using (61), we get:

$$W(2; -2; -1) = 0 \tag{70}$$

Consequently, according to (67), the reduced magnetization at the critical temperature is zero:

$$m(1) = 0 \tag{71}$$

More than this, as mentioned just before eq. (63), the magnetization is not differentiable in $t = 1$, but is still continuos. This behavior is compatible with the aspect of the experimental curve of reduced spontaneous magnetization at critical temperature (see Fig. 2).

If the magnetic field is non-zero, the definition (15) of ζ can be modified accordingly, to:

$$\zeta(t,h) = \frac{m(t,h)}{t} \tag{72}$$

So, the equation of state becomes:

$$t\zeta(t,h) = \tanh\left(\zeta(t,h) + \frac{h}{t}\right) \tag{73}$$

and we obtain for the magnetization a solution similar to (66):

$$m(t,h) = \frac{t}{2} W\left(2h + \frac{2}{t}; 2h - \frac{2}{t}; -1\right) - h \tag{74}$$

which can be written in terms of Laguerre polynomials or Lambert $r-$ function (see eq. (67)).

Taking the derivative with respect to h in both sides of (73), this transcendental equation can be put in a differential form:

$$\frac{\partial\zeta(t,h)}{\partial h} = \frac{1}{t}\frac{\left(1 - t^2\zeta^2(t,h)\right)}{\left(-1 + t + t^2\zeta^2(t,h)\right)} \tag{75}$$

and, similarly, any higher order derivative can be written in terms of t and $\zeta(t,h)$. Consequently, we can write the equation of state as a series expansion in the powers of the external magnetic field:

$$\zeta(t,h) = \zeta(t,0) + \left.\frac{\partial\zeta(t,h)}{\partial h}\right|_{h=0} h + \frac{1}{2}\left.\frac{\partial^2\zeta(t,h)}{\partial h^2}\right|_{h=0} h^2 + \ldots \tag{76}$$

where the coefficients are temperature-dependent functions, whose explicit form can be calculated, using eq. (66).

5.4. ALGEBRAIC APPROACH FOR INVERSION OF $B_J(x)$

As we already shown, we can put the Brillouin functions in an algebraic form, eq. (47); so, to obtain the inverse Brillouin function means to solve an algebraic equation. It is convenient to discuss separately the integer and half-integer case.

If $J = n$:

$$B_n(z) = \frac{1}{n}\frac{nz^{2n+2} - (n+1)z^{2n+1} + (n+1)z - n}{\left(z^{1+2n} - 1\right)(z-1)} \tag{77}$$

(the lower index of z used in eqs. (47) - (53) has been dropped, in order to avoid too complicated notations) and eq. (47) gives the following equation for $z(t)$, directly connected to the inverse Brillouin function by (53):

$$n(1-t)z^{2n+2} - (n(1-t)+1)z^{2n+1} + (n(1+t)+1)z - n(1+t) = 0\ ,\ t < 1 \tag{78}$$

It is a quite particular tetranomic equation of degree $2n+2=2(J+1)$, containing only 4 terms - namely, the two highest and the two lowest order ones; however, to the best of author's knowledge, there is no analytical formula for its roots.

The value of $z(t\to 1)$ gives the asymptotic behavior of the root, important for obtaining the asymptotic behavior of the inverse Brillouin function, anticipated in (57). We shall get it, putting $t=1$ in the coefficients of (79), excepting the dominant term (so, the substitution $t=1$ does not affect the degree of the equation) and looking for a root of the form:

$$z=\frac{a}{1-t} \tag{79}$$

The result is:

$$z_n(t\to 1)=\frac{1}{n(1-t)} \tag{80}$$

It is easy to check that the result has the same form, for integer and half-integer indices, so the asymptotic behavior of the inverse Brillouin function is:

$$B_J^{-1}(t\to 1)=J\ln z_J(t)|_{t\to 1}=J\ln\frac{1}{1-t} \tag{81}$$

The polynomial in the l.h.s. of (78) has the form:

$$(1-z)^2\sum_{k=0}^{2n}(k-(t+1)n)z^k \tag{82}$$

where the $2n-$order polynomial multiplying $(1-z)^2$ in the previous formula was firstly written by Millev and Fähnle [62, 63]. They noticed that, according to Descartes' criterion [64], it has a unique positive real root, localized in the interval $(1,\infty)$ - the root which gives the inverse Brillouin function, according to (53). The algebraic approach to inverse Brillouin functions can be used only for small indices (small values of spins), as mentioned by several authors [48, 62, 63].

Similarly, from (48), the equation whose root gives the inverse of S_J is

$$z^J+z^{J-1}+...+z^{-(J-1)}+z^{-J}=t \tag{83}$$

and its degree $2J$ can be decreased to J, putting

$$z+\frac{1}{z}=u. \tag{84}$$

Eq. (83) can be written in a more compact form:

$$z^{2n+1}-tz^{n+1}+tz^n-1=0 \tag{85}$$

It is a tetranomic equation, of degree $2n+1=2J+1$ in z, having a non-physical root in $z=1$.

If $J=n+\frac{1}{2}$, the treatment is similar, and can be found in [1].

We shall discuss the general case (47) for $J=1,\ 3/2,\ 2$ and 3.

For $J=1$, (79) gives:

$$\left((1-t)\,z^2-tz-(t+1)\right)(z-1)^2 \;=0 \tag{86}$$

So, the physical root is:

$$z\,(t)=\frac{t\pm\sqrt{t^2+4\,(1-t^2)}}{2\,(1-t)}\to\frac{t+\sqrt{4-3t^2}}{2\,(1-t)} \tag{87}$$

a result mentioned by Millev and Fähnle [62, 63] and by Kröger [48].

For $J=3/2$, the inverse Brillouin function is a root $z\,(t)$ of the equation:

$$3\,(1-t)\,z^3+(1-3t)\,z^2-(1+3t)\,z-3\,(1+t)=0 \tag{88}$$

Its exact solution is given in terms of algebraic functions in [63] and in terms of hyperbolic functions in [1].

As shown in [1], $S_{3/2}^{-1}\,(t)$ is given by the root of the equation:

$$S_{\frac{3}{2}}\,(z)=z^{\frac{3}{2}}+z^{\frac{1}{2}}+z^{-\frac{1}{2}}+z^{-\frac{3}{2}}=t \tag{89}$$

With a change of variable

$$z^{\frac{1}{2}}+z^{-\frac{1}{2}}=u \tag{90}$$

it becomes a cubic trinomial equation, with the physically convenient root having a compact form, in terms of trigonometric functions (see [1], eq. (119)].

For $J=2$, the inverse B_2^{-1} is obtained as a root of a quartic equation (see [1], eq. (119)), but S_2^{-1} can be obtained solving a second order equation in $z+\frac{1}{z}$.

For $J=3$, the tetranomic equation which gives B_3^{-1} is sextic, so practically unsolvable, but that for S_3^{-1} is a quite simple cubic equation.

Consequently, for $J=2$ and $J=3$, it could be useful to obtain S_J^{-1} and then B_J^{-1}, using (42).

6. APPROXIMATE RESULTS CONCERNING THE INVERSION OF $B_J\,(x)\,,\;L\,(x)\,,\;B_J\,(x)\,/x,\;L\,(x)\,/x$

6.1. THE FUNCTIONS $L\,(x)$

There are several recent papers on the approximants of the inverse Langevin functions, including excellent reviews, some of them already cited [11–13]. In [65] is discussed (and plotted, see Fig. 4) the Taylor expansion of L^{-1}, with 115 terms: it is very precise in the interval $[0,0.95]$, but very poor outside. This situation is illustrative for the need of subtle analytic approximation for such functions, or interesting as obtained by a diversity of approaches. As the reviews just cited are almost exhaustive, we shall mention here some few results, which are more relevant, or less known. In [11], Jedynak obtains a very precise rounded Padé approximation:

$$L^{-1}(y) = y\frac{3-2.6y+0.7y^2}{1-0.9y-0.1y^2} = y\frac{3-2.6y+0.7y^2}{(1-y)(1+0.1y)} \tag{91}$$

One of the most accurate approximations is that of Bergström (1999), eq. (11) of [11]:

$$L^{-1}(0<y<0.84136) = 1.31446\tan(1.58986y)+0.91209y \tag{92}$$

$$L^{-1}(0.84136<y<1) = \frac{1}{1-y} \tag{93}$$

However, it is not easily applicable for physical models, as it is defined by two different mathematical expressions, on two subintervals.

Keady also proposed an interesting approximation for the inverse Langevin function [27]:

$$L^{-1}(y) \simeq \frac{6}{\pi}\tau\left(\frac{1+b_n\tau^2}{1+b_d\tau^2}\right) \tag{94}$$

with

$$\tau = \tan\left(\frac{\pi y}{2}\right),\ b_n = \frac{\pi^2}{12}b_d,\ b_d = \frac{20\pi^2-144}{\pi^2(60-5\pi^2)} \simeq 0.508 \tag{95}$$

Another algebraic-trigonometric formula was given by Petrosyan [10]:

$$L^{-1}(y) = 3y+\frac{y^2}{5}\sin\frac{7y}{2}+\frac{y^3}{1-y} \tag{96}$$

6.2. THE FUNCTIONS $B_J(x)$ AND $B_J(x)/x$

There are several approximations for the inverse Brillouin functions, intended to be appropriate for teaching, for a specific theory (e.g. hysteretic physics), for analytic approximation of certain results, etc.

A popular approximation useful for teaching was proposed by Arrott [3], who noticed that the expression:

$$B(x) = \frac{x}{\sqrt{x^2+a^2}} = \frac{1}{a}x+O\left(x^3\right) \tag{97}$$

behaves similarly to $B_J(x)$, if $a = (J+1)/3J$:

$$B_J(x) \simeq \frac{x}{\sqrt{x^2+a_J^2}},\ a_J = \frac{J+1}{3J} \tag{98}$$

It can be easily inverted, so:

$$B_J^{-1}(x) \simeq \frac{a_J x}{\sqrt{1-x^2}} \tag{99}$$

The approximation is really excellent for $J=4$:

$$B_4(x)\simeq\frac{x}{\sqrt{x^2+\left(\frac{12}{5}\right)^2}} \tag{100}$$

(in eq. (9) of [3], it is written, incorrectly, $5/12$ instead of $12/5$), when the error ε is between -0.65 and 0.3%. For $J\neq 4$, the error is somewhat larger, $\sim 1\%$. Unfortunately, (105) describes incorrectly the singularity of $B_J^{-1}(x)$, which is not $\frac{1}{\sqrt{1-x}}$, but $\ln\frac{1}{1-x}$.

In the context of hysteretic physics, Takacs [66] proposed a very simple approximation, which can be easily inverted easily:

$$B_J(x)\simeq\tanh(d_J x) \tag{101}$$

with

$$d_J=\frac{1}{2.667J}+0.25 \tag{102}$$

but the errors are $3.4-6.3\%$ $(J=1)$; $7-10\%$ $(J=3/2)$; $9.3-12.5\%$ $(J=2)$; $10-14\%$ $(J=5/2)$; $13-15\%$ $(J=3)$; $13.5-16.6\%$ $(J=7/2)$, so much larger than Arrott's ones. However, the function obtained by inverting (105) have the correct asymptotic behavior.

Also, Takacs claims that the characteristic curves of ferromagnetism (mainly, of hysteretic physics) can be deduced from and described by a simple combination of linear and hyperbolic functions [67]:

$$T(x)=A_0x+B_0\tanh C_0x \tag{103}$$

and the complexity of an analytic theory makes compulsory the use of very severe approximations for B_J and B_J^{-1}. It is an elementary exercise to show that the inverse of $T(x)$ is a generalized Lambert function, and its explicit form is given by (60), but this formula seems to be of limited practical use.

In a recent paper [48], Kröger developed a comprehensive analysis of the inverse Langevin and Brillouin functions. One of his results is the following:

$$B_J^{-1}(t)=\frac{1}{2}\frac{15-11(1-\varepsilon)(1+2\varepsilon)t^2}{5+10\varepsilon-(1-\varepsilon)[5+11\varepsilon(1+2\varepsilon)]t^2}\ln\frac{1+t}{1-t},\ \ \varepsilon=\frac{1}{2J} \tag{104}$$

This is a very simple and precise formula, satisfying the symmetry property of B_J^{-1}, eq. (29). Its accuracy increases for larger indices (maximum relative errors from 1.5% for $J=3/2$ to 0.6% for $J=5$ and to 0.35% for $J=10$).

An alternative formula was proposed recently:

$$B_J^{-1}(t>0)=\frac{3J}{1+J}P_J(t)\ln\left(\frac{1}{1-t}\right),\ B_J^{-1}(t<0)=-B_J^{-1}(-t) \tag{105}$$

with the polynomial $P_J(t)$ satisfying the conditions

$$P_J(0) = 1,\ P_J(1) = \frac{1+J}{3} \tag{106}$$

Its degree is not specified *ab initio*, it can be chosen accordingly to the specificity of the problem under investigation. Its explicit form can be obtained elementarily, choosing as many $(t_j, P_J(t_j))$ numerical pairs as we want, and using the *Fitting curves to data* command in MATHEMATICA.

Actually, it is a purpose-built approximant: it gives a simple recipe to find an approximant, with the simplest mathematical form, with tunable accuracy. With a sufficiently large number of points $(t_j, P_J(t_j))$, we can obtain a sufficiently high accuracy for the coefficients, according to our specific goal. However, this approach has two disadvantages: (1) the specific expression of the polynomial $P_J(y)$ changes if the distribution of points $(y_j, P_J(y_j))$ changes and (2) the degree of the polynomial which grants a good accuracy is too large.

A more elegant and efficient approach was advanced by Kröger [68] and included in [69]. Kröger proposed a code which generates an arbitrary large number of intermediary points t_j; it was applied by Tolea [70] to the calculation of the polynomial $P_J(t)$, with the following results: (1) if the number of intermediate points is about 10^6, the first 6 digits of the coefficients of $P_J(t)$ do not change if this number increases and (2) a polynomial as simple as a sextic one is sufficient to obtain an approximant with an accuracy better than any of the approximants currently in use, and better than the accuracy of most experiments in ferromagnetism. For instance, for $J = 2$, Tolea obtains the following result:

$$P_2^{(6)}(x) = 1 - 0.5965867x + 1.848608x^2 - 8.032131x^3 + \\ + 18.04970x^4 - 18.80268x^5 + 7.398635x^6 \tag{107}$$

with an error of about 10^{-4} $\left(-2\cdot 10^{-4} \lesssim \varepsilon \lesssim 1.5\cdot 10^{-4}\right)$.

For the inversion of functions $B_J(x)/x$, we can proceed similarly. However, if the goal of obtaining the inverse of $B_J(x)/x$ is to find the explicit temperature dependence of magnetization in a mean-field model, as explained in Section 2, is simpler to circumvent this approach, and to look for an approximant having the form:

$$m_J(t) = Q_J(t)\sqrt{1-t} \tag{108}$$

where $Q_J(t)$ is a polynomial, whose degree is given by the number of interpolation points. Five polynomials of degree 7, $Q_J(t)$, for $J = 1/2, 1, \ldots, 5/2$, with relative deviations of about 10^{-7}, are given in [59].

7. CONCLUSIONS

In this paper, we presented the results concerning mainly the inverses of $L(x)$, $B_J(x)$, $L(x)/x$ and $B_J(x)/x$, obtained recently by researchers working in several domains of physics – rubber elasticity, solar energy conversion, ferromagnetism, hysteretic physics – and of mathematics – theory of transcendental or algebraic equations –, and also added a few others, new and yet unpublished. This review might be useful, as researchers working in so diverse fields are not necessarily aware of the progress registered by their colleagues, focused on different physical problems, which share, however, a common mathematical basis. Actually, the increase of accuracy of experimental investigations, on one side, and the recent results of the theory of generalized Lambert functions, on the other, fostered the interest for the functions discussed in this review.

The exact results enlarge the list of thermodynamical quantities to be theoretically evaluated in exactly solvable models, like mean field models in magnetism or ferrofluids. With the approximate analytical formulas, results available till now only numerically, for instance in the Callen - Shtrikman program, or in the Harrison positive-feedback model, can be put in an analytical form. Another new field of research could be the applications of inverse Langevin function in phenomena described by doubly-truncated exponential distribution, as Keady pointed out.

Whenever possible, the pedagogical potential of the approximate results was underlined. The ways to improve the accuracy of analytical approximations of inverse functions studied in this paper are also discussed.

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