Romanian Reports in Physics **72**, 110 (2020)

PAINLEVÉ ANALYSIS FOR HIGHER–DIMENSIONAL INTEGRABLE SHALLOW WATER WAVES EQUATIONS WITH TIME–DEPENDENT COEFFICIENTS

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Received January 3, 2020

Abstract. We investigate the (2+1)- and (3+1)-dimensional shallow water waves equations with time-dependent coefficients. We study the integrability of each developed model by using the Painlevé approach. Multiple soliton solutions and multiple complex soliton solutions, for each equation, are constructed using the Hirota's direct method, and the related complex criteria.

Key words: Painlevé analysis, compatibility conditions, shallow water waves equations.

1. INTRODUCTION

Nonlinear integrable equations have attracted a lot of attention and many powerful algorithms have been introduced for investigating these nonlinear partial differential equation [1–28]. Research works on equations with time-dependent coefficients are gaining more attention because of their close relation to physical phenomena and engineering applications.

The field of shallow water waves has significant influence on the ocean engineering, marine environment, atmospheric science, and basically on the planet's climate. The equations of shallow water waves describe the motion of water forms wherein the depth is short corresponding to the scale of the waves propagating on that form [1–3].

Three well-known shallow water waves equations in (1+1) dimensions, namely

$$u_t + u_x - 3u_x u_t - u_{xxt} = 0, \tag{1}$$

$$u_t + 3(u_x)^2 - 3u_x u_t - u_{xxx} - u_{xxt} = 0, \tag{2}$$

and

$$u_t + u_x + 3(u_x)^2 - 3u_x u_t + u_{xxx} - u_{xxt} = 0, \tag{3}$$

have been studied with many useful findings [1–3]. Moreover, higher dimensional shallow water waves equations in (2+1) and (3+1) dimensions, given as

$$u_{yt} - 3u_x u_{xy} - 3u_{xx}u_y + u_{xxxy} - u_{xy} = 0, \qquad (4)$$

and

$$u_{yt} - 3u_x u_{xy} - 3u_{xx}u_y + u_{xxxy} - u_{yz} = 0, \qquad (5)$$

respectively, were also extensively studied with many useful results. Notice that the only difference between Eq. (4) and (5) relies on the last term of each equation. It is worth noting that the five above-mentioned equations have constant coefficients and all are confirmed to be integrable through passing the Painlevé test. Moreover, the first three equations (1)–(3) with time-dependent coefficients were investigated in Ref. [2]. The shallow water wave equations are used in wide applications, such as ocean engineering, coastal engineering, tsunami prediction, tidal waves, river and irrigation flows, weather simulations, and other applications as well. To the best of our knowledge, these equations, with time-dependent coefficients that will be examined in this paper, have not been obtained and investigated before.

The studies of entirely integrable models are thriving as they describe a series of phenomena in physics and engineering. Various effective procedures were built in the literature to construct integrable models and to assort dynamical natures of the developed integrable models. The strategy for the coming Sections will be as follows. We will first develop and study the (2+1)- and (3+1)-dimensional shallow water wave models (4) and (5), but with time-dependent coefficients instead of constant coefficients. We will use the Painlevé analysis, to examine the conditions on the time-dependent coefficients that will confirm the complete integrability to each of the examined model. We will formally derive multiple real and multiple complex soliton solutions upon using the simplified Hirota's method and the related complex criteria.

2. THE (2+1)-DIMENSIONAL SHALLOW WATER MODEL WITH TIME-DEPENDENT COEFFICIENTS

The (2+1)-dimensional shallow water model with time-dependent coefficients reads

$$f_1(t)u_{yt} + f_2(t)u_x u_{xy} + f_3(t)u_{xx}u_y + f_4(t)u_{xxxy} + f_5(t)u_{xy} = 0, \qquad (6)$$

where $f_i(t) \neq 0, 1 \leq i \leq 5$. As stated earlier, we will use the Painlevé analysis to find the compatibility condition to ensure the integrability of the first model (6).

2.1. PAINLEVÉ ANALYSIS

First, we perform the singular manifold analysis for equation (6) and confirm its Painlevé integrability. We assume that equation (6) has a solution as a Laurent expansion about a singular manifold $\psi = \psi(x,y,t)$ as

$$u(x,y,t) = \sum_{k=0}^{\infty} u_k(x,y,t)\psi^{k-\gamma}, \tag{7}$$

where $u_k(x,y,t)$ $(k = 0,1,2,...)$ are functions of x, y, and t. On substitution of (7) in equation (6), and using the Painlevé analysis, we find four resonance points that occur at $k = -1,1,4$, and 6. After computations, we found the explicit solutions for u_2, u_3, and u_5. Moreover, we found that the compatibility condition to ensure integrability requires that $f_1(t) = f(t), f_2(t) = f_3(t) = f_4(t) = g(t)$, and $f_5(t) = h(t)$, where $f(t), g(t)$, and $h(t)$ remain any non-zero differentiable functions. We should verify the compatibility conditions at the resonance points $k = 1,4$, and 6. For this purpose, the Laurent series may be truncated at the maximum resonance point as

$$u(x,y,t) = \sum_{k=0}^{6} u_k(x,y,t)\psi^{k-\gamma}. \tag{8}$$

This in turn gives u_1, u_4, and u_6 to be arbitrary functions and this confirms the complete integrability of this nonlinear model.

2.2. MULTIPLE SOLITON SOLUTIONS FOR THE (2+1)-DIMENSIONAL EQUATION

Based on the compatibility condition derived warlier, the (2+1)-dimensional shallow water model with time-dependent coefficients (6) becomes

$$f(t)u_{yt} + g(t)u_x u_{xy} + g(t)u_{xx}u_y + g(t)u_{xxxy} + h(t)u_{xy} = 0. \tag{9}$$

or equivalently

$$f(t)u_{yt} + g(t)(u_x u_y + u_{xxy})_x + h(t)u_{xy} = 0. \tag{10}$$

We first substitute

$$u(x,y,t) = e^{k_i x + r_i y - \omega_i(t)}, \tag{11}$$

into the linear terms of (10), and we find that the dispersion $\omega_i(t)$ is

$$\omega_i(t) = \int \frac{k_i^3 g(t) + k_i h(t)}{f(t)} dt, i = 1,2,3. \tag{12}$$

Consequently, the phase variables are given as

$$\theta_i = k_i x + r_i y - \omega_i(t), i = 1,2,3. \tag{13}$$

We next use the transformation

$$u(x,y,t) = 6(\ln F(x,y,t))_x, \tag{14}$$

where the auxiliary function $F(x,y,t)$ for the single soliton solution is given by

$$F(x,y,t) = 1 + e^{\theta_1} = 1 + e^{k_1x + r_1y - \omega_1(t)}, \tag{15}$$

to obtain the single soliton solution as

$$u(x,y,t) = \frac{6k_1 e^{k_1x + r_1y - \omega_1(t)}}{1 + e^{k_1x + r_1y - \omega_1(t)}}. \tag{16}$$

The auxiliary function

$$F(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \tag{17}$$

will allow us to to determine the two-soliton solutions, where the phase variables $\theta_i, i = 1,2,3$ are given earlier in (13), and a_{12} is the phase shift that we determine as

$$a_{12} = \frac{(k_1 - k_2)(r_1 - r_2)}{(k_1 + k_2)(r_1 + r_2)}, \tag{18}$$

which can be generalized to

$$a_{ij} = \frac{(k_i - k_j)(r_i - r_j)}{(k_i + k_j)(r_i + r_j)}, 1 \leq i < j \leq 3. \tag{19}$$

Substituting (17)–(18) into (14) gives the two-soliton solutions.

For the three-soliton solutions, we set the auxiliary function by

$$\begin{aligned} f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} \\ + a_{12}a_{23}a_{13}e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{20}$$

The three-soliton solutions are readily obtained, and this gives N-soliton solutions for finite N, where $N \geq 1$.

2.3. MULTIPLE COMPLEX SOLITON SOLUTIONS FOR THE (2+1)-DIMENSIONAL EQUATION

In this Section, we will apply the complex criteria [1] of the simplified Hirota's method to the equation (10).

It is interesting to note that the dispersion relations and the dependent variable transformation for this equation are the same as obtained before and are given as

$$\begin{aligned} \omega_i &= \omega_i(t) = \int \tfrac{k_i^3 g(t) + k_i h(t)}{f(t)} dt, i = 1,2,3, \\ u(x,y,t) &= 6(\ln F(x,y,t))_x. \end{aligned} \tag{21}$$

For single complex soliton solution we use the auxiliary complex function as

$$F(x,y,t) = I + e^{k_1x + r_1y - \omega_1(t)}, I = \sqrt{-1}, \tag{22}$$

and we obtain

$$u(x,y,t) = \frac{6Ik_1 e^{k_1x + r_1y - \omega_1(t)}}{I + e^{k_1x + r_1y - \omega_1(t)}}. \tag{23}$$

For two complex soliton solutions we use the auxiliary complex function as

$$F(x,y,t) = I + e^{k_1 x-\omega_1(t)} + e^{k_2 x-\omega_2(t)} - I a_{12} e^{(k_1+k_2)x-(\omega_1(t)+\omega_2(t))}, \tag{24}$$

where we find that the phase shift is the same as derived earlier for the real case. Substituting these results into (14) gives the two complex soliton solutions.

For the three complex soliton solutions we use the auxiliary complex function as

$$\begin{aligned} F(x,y,t) = I + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - I a_{12} e^{\theta_1+\theta_2} - I a_{13} e^{\theta_1+\theta_3} - I a_{23} e^{\theta_2+\theta_3} \\ - a_{12} a_{23} a_{13} e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{25}$$

3. THE (3+1)-DIMENSIONAL SHALLOW WATER MODEL WITH TIME-DEPENDENT COEFFICIENTS

The (3+1)-dimensional shallow water model with time-dependent coefficients reads

$$f_1(t)u_{yt} + f_2(t)u_x u_{xy} + f_3(t)u_{xx}u_y + f_4(t)u_{xxxy} + f_5(t)u_{yz} = 0, \tag{26}$$

where $f_i(t) \neq 0, 1 \leq i \leq 5$. As stated earlier, we will use the Painlevé analysis to find the compatibility condition to ensure the integrability of the model (26).

3.1. PAINLEVÉ ANALYSIS

Proceeding as before, we find four resonance points that occur at $k = -1, 1, 4,$ and 6. The compatibility condition to ensure integrability requires that $f_1(t) = f(t), f_2(t) = f_3(t) = f_4(t) = g(t)$, and $f_5(t) = h(t)$, where $f(t), g(t)$, and $h(t)$ remain any non-zero differentiable functions.

3.2. MULTIPLE SOLITON SOLUTIONS FOR THE (3+1)-DIMENSIONAL EQUATION

Based on the compatibility condition we derived before, the (3+1)-dimensional shallow water model with time-dependent coefficients (26) becomes

$$f(t)u_{yt} + g(t)u_x u_{xy} + g(t)u_{xx}u_y + g(t)u_{xxxy} + h(t)u_{yz} = 0. \tag{27}$$

or equivalently

$$f(t)u_{yt} + g(t)(u_x u_y + u_{xxy})_x + h(t)u_{yz} = 0. \tag{28}$$

We first substitute

$$u(x,y,z,t) = e^{k_i x + r_i y + s_i z - \mu_i(t)}, \tag{29}$$

into the linear terms of (28), and we find that the dispersion $\mu_i(t)$ is

$$\mu_i(t) = \int \frac{k_i^3 g(t) + s_i h(t)}{f(t)} dt, i = 1, 2, 3. \tag{30}$$

For simplicity. we proceed as above to obtain the single-soliton solution as

$$u(x,y,z,t) = \frac{6k_1 e^{k_1x+r_1y+s_1z-\mu_1(t)}}{1+e^{k_1x+r_1y+s_1z-\mu_1(t)}}. \tag{31}$$

For the two-soliton and three-soliton solutions, we proceed as above to get the same phase shift a_{ij} as obtained for the (2+1)-dimensional case.

4. CONCLUSION

Two higher dimensional shallow water waves models with time-dependent coefficients were investigated in this paper. The Painlevé analysis was employed for each of the two models to derive the compatibility conditions to ensure the integrability of each model. Multiple soliton solutions and multiple complex soliton solutions were systematically constructed using the Hirota's method and its complex companion.

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