

## SOLITONS OF THE COMPLEX NONLINEAR SCHRÖDINGER EQUATION WITH PARITY-TIME-SYMMETRIC LINEAR-NONLINEAR LATTICE POTENTIALS

SYED TAHIR RAZA RIZVI<sup>1</sup>, UJALA SAFDAR<sup>1</sup>, ISHRAT BIBI<sup>1</sup>, KASHIF ALI<sup>1</sup>, QIN ZHOU<sup>2,\*</sup>

<sup>1</sup>Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan

<sup>2</sup>School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan 430212, China

\*Corresponding author, Email: qinzhou@whu.edu.cn

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*Abstract.* By using an inverse method, we report families of exact soliton solutions of a generalized complex nonlinear Schrödinger equation with parity-time-symmetric mixed linear-nonlinear lattice potentials. The Kerr-, parabolic-, and power-law nonlinearities are considered and analytical soliton solutions are obtained, for some special forms of complex-valued parity-time-symmetric potentials.

*Key words:* optical solitons, PT-symmetric complex potentials, mixed linear-nonlinear optical lattices, inverse method.

### 1. INTRODUCTION

Optical lattice solitons form as a result of precise balance of lattice modulation, linear diffraction, and nonlinear effects. In a pioneering work published in 2003, Fleischer *et al.* [1] used light induced techniques to create nonlinear periodic lattices and observed two-dimensional discrete solitons in optically induced nonlinear photonic lattices. We also refer here to earlier pioneering works on discrete solitons in a series of physical settings: studies of solitary excitons in one-dimensional molecular chains [2], solitons in polyacetylene [3], discrete solitons in nonlinear arrays of coupled optical waveguides [4], and discrete solitons and breathers in dilute Bose-Einstein condensates [5].

In a seminal paper published in 1998, Bender and Boettcher [6] have shown that non-Hermitian Hamiltonians can exhibit entirely real spectra if they obey parity-time (PT)-symmetry. Thus the concept of PT-symmetric complex-valued external potentials has been put forward and the studies of optical lattice solitons have been shifted from the real domain to the complex one, see two comprehensive review papers on solitons and nonlinear waves that form in nonlinear lattices and PT-symmetric physical systems [7, 8]. In an optical lattice the complex-valued refractive index is  $q(x) = q_R(x) + iq_I(x)$ , and it plays the role of an external complex potential. We obtain a PT-symmetric complex-valued potential if the real part  $q_R(x) = q_R(-x)$  is

an even function on the spatial coordinate  $x$  and the imaginary part  $q_I(x) = -q_I(x)$  is an odd function; see, for example, the works [9–14] for studies of solitons in PT-symmetric optical lattices and other PT-symmetric physical systems.

The generic physical model for the PT-symmetric lattice solitons is the complex nonlinear Schrödinger equation (NLSE). Many types of solitons in PT-symmetric physical settings have been reported, including gap solitons, defect solitons, multi-lump solitons, etc. [15–39]; for two comprehensive recent reviews on nonlinear waves in optical and matter-wave media, including recent studies of solitary waves in various PT-symmetric physical systems, see Refs. [40, 41].

In this work, we will expand the regular complex NLSE to a more general nonlinear evolution equation, the so-called integer order NLSE. We obtain the exact soliton solutions and the corresponding PT-symmetric mixed linear-nonlinear lattice potentials for the integer order NLSE. The basic tool is an inverse method, which is also called the *inverse engineering method*, see Refs. [28, 29]. We also refer here to recent works on the existence and dynamics of solitons in media with imprinted PT-symmetric mixed linear-nonlinear optical lattices [42–47].

The general dynamical model for optical solitons that form in PT-symmetric mixed linear-nonlinear lattice potentials is given by [45]:

$$iu_z^n + \frac{1}{2}u_{xx}^n + D_l(x)u^n + D_{nl}(x)|u|^2u^n + F(|u|^2)u^n = 0. \quad (1)$$

In Eq. (1), the complex field amplitude is represented by  $u(x, z)$ , where  $n$  is an integer number. The complex-valued potentials, which are characterized by space-depending parameters, are  $D_l(x) = U_1(x) + iV_1(x)$  and  $D_{nl}(x) = U_2(x) + iV_2(x)$ , respectively. The term  $D_l(x)$  represents the PT-symmetric linear lattice potential, while  $D_{nl}$  gives the nonlinear complex potential.

For PT-symmetric potentials, we have:

- The real parts should be even functions, that is:  $U_l(x) = U_l(-x)$  for  $l = 1, 2$ .
- The imaginary parts should be odd functions, that is:  $V_l(x) = -V_l(-x)$  for  $l = 1, 2$ .

## 2. THEORETICAL ANALYSIS

By using an inverse method (the so-called, inverse engineering method), we will obtain families of exact soliton solutions for Eq. (1) with the following ansatz.

$$u(x, z) = R(x)e^{\Upsilon}, \quad (2)$$

where

$$\Upsilon = i\left(\rho z + \int g(x)dx\right). \quad (3)$$

In Eq. (2), the real amplitude is represented by  $R(x)$ , the inhomogeneous phase of the optical mode is given by  $g(x)$ , while  $\rho$  is the nonlinear propagation constant. Now by insering Eq. (2) into Eq. (1), we obtain the real and imaginary parts given as:

$$\frac{1}{2}nR\frac{d^2R}{dx^2} + \frac{1}{2}n(n-1)\left(\frac{dR}{dx}\right)^2 - \left[\frac{1}{2}n^2g^2(x) + n\rho - U_1(x)\right]R^2 + U_2(x)R^4 + F(R^2)R^2 = 0, \quad (4)$$

and

$$g(x) = -\frac{2}{R^{2n}} \int (R^{2n}[V_2(x)R^2 + V_1(x)])dx. \quad (5)$$

We take the integration constant in Eq. (4) to be zero. Thus after using the following steps, we can easily obtain the exact solitons solution for Eq. (1):

- **Step 1.** Remove the phase gradient  $g(x)$  from Eq. (4) for known functions  $V_1(x)$ ,  $V_2(x)$ ,  $R(x)$ .
- We choose  $R(x) = R_0 \operatorname{sech}(x)$  to get bright solitons,  $R(x) = R_0 \operatorname{tanh}(x)$  to obtain dark solitons, and  $R(x) = R_0 \operatorname{coth}(x)$  to get singular solitons.
- The imaginary parts  $V_1(x)$  and  $V_2(x)$  of the PT-symmetric potentials must be odd functions.
- **Step 2.** By using the values of  $g(x)$  and  $R(x)$  in Eq. (3), we build a relationship between  $U_1(x)$  and  $U_2(x)$  and then find their exact expressions.
- The real components  $U_1(x)$  and  $U_2(x)$  must be even functions.

## 2.1. KERR-LAW NONLINEARITY

In Kerr-law nonlinear media, we have  $F(R^2) = R^2$ . Thus Eq. (4) reduces to

$$\frac{1}{2}nR\frac{d^2R}{dx^2} + \frac{1}{2}n(n-1)\left(\frac{dR}{dx}\right)^2 - \left[\frac{1}{2}n^2g^2(x) + n\rho - U_1(x)\right]R^2 + U_2(x)R^4 + R^4 = 0. \quad (6)$$

**Case 1:** We presume the following expressions of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (7)$$

The amplitudes of complex-valued external potentials are represented by  $\beta_1$  and  $\beta_2$ . For bright solitons, we consider  $R(x) = R_0 \operatorname{sech}(x)$ :

$$u(x, z) = R_0 \operatorname{sech}(x) e^{\Upsilon}, \quad (8)$$

and the potentials are given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2U_2 \right] \operatorname{sech}^2(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \sinh^3(x), \quad (9)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \cosh^{2n}(x) \sinh^3(x), \quad (10)$$

Here we have

$$g(x) = -2\beta_2 R_0^2 \cosh^{2n}(x) \cosh(x) - 2\beta_2 R_0^2 \cosh^{2n}(x) \operatorname{sech}(x) + \frac{3}{2}\beta_1 \cosh^{2n}(x) \cosh(x) - \frac{1}{6}\beta_1 \cosh^{2n}(x) \cosh(3x). \quad (11)$$

**Case 2:** We suppose the following expressions of  $V_1(x)$  and  $V_2(x)$

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (12)$$

We choose  $R(x) = R_0 \tanh(x)$  for dark soliton solutions:

$$u(x, z) = R_0 \tanh(x) e^{\Upsilon}, \quad (13)$$

The potential is given by

$$D_l(x) = - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} + \frac{1}{2}n^2g^2(x) + i\beta_1 \coth^{2n}(x) \sinh^3(x), \quad (14)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \coth^{2n}(x) \sinh^3(x), \quad (15)$$

where

$$g(x) = \frac{7}{2}\beta_2 R_0^2 \coth^{2n}(x) \cosh(x) - \frac{1}{6}\beta_2 R_0^2 \coth^{2n}(x) \cosh(3x) + 2\beta_2 R_0^2 \coth^{2n}(x) \operatorname{sech}(x) + \frac{3}{2}\beta_1 \coth^{2n}(x) \cosh(x) - \frac{1}{6}\beta_1 \coth^{2n}(x) \cosh(3x). \quad (16)$$

**Case 3:** We assume the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \tanh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \tanh^{2n}(x) \sinh^3(x). \quad (17)$$

We consider  $R(x) = R_0 \coth(x)$  for singular soliton solutions:

$$u(x, z) = R_0 \coth(x) e^{\Upsilon}, \quad (18)$$

and the potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2 g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2 U_2 \right] \coth^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\coth^2(x)} \right) + i\beta_1 \tanh^{2n}(x) \sinh^3(x), \quad (19)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \tanh^{2n}(x) \sinh^3(x), \quad (20)$$

where

$$g(x) = -\frac{2}{3}\beta_2 R_0^2 \tanh^{2n}(x) \cosh^3(x) + \frac{3}{2}\beta_1 \tanh^{2n}(x) \cosh(x) - \frac{1}{6}\beta_1 \tanh^{2n}(x) \cosh(3x). \quad (21)$$

**Case 4:** We use the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (22)$$

For bright soliton solutions, we consider  $R(x) = R_0 \operatorname{sech}(x)$ :

$$u(x, z) = R_0 \operatorname{sech}(x) e^{\Upsilon}, \quad (23)$$

and the corresponding potential is given by

$$D_l(x) = \frac{1}{2}n^2 g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2 U_2 \right] \operatorname{sech}^2(x) + n\rho - \frac{1}{2}n^2 + i\beta_1 \cosh^{2n}(x) \tanh^3(x), \quad (24)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \cosh^{2n}(x) \tanh^3(x), \quad (25)$$

where

$$g(x) = -\frac{1}{2}\beta_2 R_0^2 \cosh^{2n}(x) \tanh^4(x) - 2\beta_1 \cosh^{2n}(x) \ln(\cosh(x)) - \beta_1 \cosh^{2n}(x) \operatorname{sech}^2(x). \quad (26)$$

**Case 5:** We presume the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (27)$$

For dark soliton solutions, we take  $R(x) = R_0 \tanh(x)$

$$u(x, z) = R_0 \tanh(x) e^{\Upsilon}, \quad (28)$$

and the corresponding potential is obtained as

$$D_l(x) = \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} + i\beta_1 \coth^{2n}(x) \tanh^3(x), \quad (29)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \coth^{2n}(x) \tanh^3(x), \quad (30)$$

where

$$g(x) = -2\beta_2R_0^2 \coth^{2n}(x) \ln(\cosh(x)) - 2\beta_2R_0^2 \coth^{2n}(x) \operatorname{sech}^2(x) + \frac{1}{2}\beta_2R_0^2 \coth^{2n}(x) \operatorname{sech}^4(x) - 2\beta_1 \coth^{2n}(x) \ln(\cosh(x)) - 2\beta_1 \coth^{2n}(x) \operatorname{sech}^2(x). \quad (31)$$

**Case 6:** Here we use the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \tanh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \tanh^{2n}(x) \tanh^3(x). \quad (32)$$

We take  $R(x) = R_0 \coth(x)$  for singular soliton solution:

$$u(x, z) = R_0 \coth(x) e^{\Upsilon}, \quad (33)$$

and the associated potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \coth^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\coth^2(x)} \right) + i\beta_1 \tanh^{2n}(x) \tanh^3(x), \quad (34)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \tanh^{2n}(x) \tanh^3(x), \quad (35)$$

where

$$g(x) = -2\beta_2R_0^2 \tanh^{2n}(x) \ln(\cosh(x)) - 2\beta_1 \tanh^{2n}(x) \ln(\cosh(x)) - \beta_1 \tanh^{2n}(x) \operatorname{sech}^2(x). \quad (36)$$

**Case 7:** Here we use the following expressions of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (37)$$

For bright soliton solution, we consider  $R(x) = R_0 \operatorname{sech}(x)$ :

$$u(x, z) = R_0 \operatorname{sech}(x) e^{\Upsilon}, \quad (38)$$

thus the potentials can be written as

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2U_2 \right] \operatorname{sech}^2(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \tanh^3(x), \quad (39)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \cosh^{2n}(x) \sinh^3(x), \quad (40)$$

where

$$g(x) = -2\beta_2 R_0^2 \cosh^{2n}(x) (\cosh(x) + \operatorname{sech}(x)) - 2\beta_1 \cosh^{2n}(x) \ln(\cosh(x)) - \beta_1 \cosh^{2n}(x) \operatorname{sech}^2(x). \quad (41)$$

**Case 8:** We assume the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (42)$$

We take  $R(x) = R_0 \tanh(x)$  for dark soliton solution:

$$u(x, z) = R_0 \tanh(x) e^{\Upsilon}, \quad (43)$$

and the potential is given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) - \left[ \frac{1}{2} n(n+1) + R_0^2 + R_0^2 U_2 \right] \tanh^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \tanh^3(x), \quad (44)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \coth^{2n}(x) \sinh^3(x), \quad (45)$$

where

$$g(x) = \frac{7}{2} \beta_2 R_0^2 \coth^{2n}(x) \cosh(x) - \frac{1}{6} \beta_2 R_0^2 \coth^{2n}(x) \cosh(3x) - 2\beta_2 R_0^2 \coth^{2n}(x) \operatorname{sech}(x) - 2\beta_1 \coth^{2n}(x) \ln(\cosh(x)) - \beta_1 \coth^{2n}(x) \operatorname{sech}^2(x). \quad (46)$$

**Case 9:** We assume the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \tanh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \tanh^{2n}(x) \sinh^3(x). \quad (47)$$

We choose  $R(x) = R_0 \coth(x)$  for singular soliton solution:

$$u(x, z) = R_0 \coth(x) e^{\Upsilon}, \quad (48)$$

and finally the potential is given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) - \left[ \frac{1}{2} n(n+1) + R_0^2 + R_0^2 U_2 \right] \coth^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\coth^2(x)} \right) + i\beta_1 \tanh^{2n}(x) \tanh^3(x), \quad (49)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \tanh^{2n}(x) \sinh^3(x), \quad (50)$$

where

$$g(x) = -\frac{2}{3}\beta_2 R_0^2 \tanh^{2n}(x) \cosh^3(x) - 2\beta_1 \tanh^{2n}(x) \ln(\cosh(x)) - \beta_1 \tanh^{2n}(x) \operatorname{sech}^2(x). \quad (51)$$

**Case 10:** We take the following expressions of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (52)$$

We consider  $R(x) = R_0 \operatorname{sech}(x)$  for bright soliton solution:

$$u(x, z) = R_0 \operatorname{sech}(x) e^{\Upsilon}, \quad (53)$$

and the complex-valued potential is given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) + \left[ \frac{1}{2} n(n+1) - R_0^2 - R_0^2 U_2 \right] \operatorname{sech}^2(x) + n\rho - \frac{1}{2} n^2 \right) + i\beta_1 \cosh^{2n}(x) \sinh^3(x), \quad (54)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \cosh^{2n}(x) \tanh^3(x), \quad (55)$$

where

$$g(x) = -\frac{1}{2}\beta_2 R_0^2 \cosh^{2n}(x) \tanh^4(x) + \frac{3}{4}\beta_1 \cosh^{2n}(x) \cosh(x) - \frac{1}{6}\beta_1 \cosh^{2n}(x) \cosh(3x) \quad (56)$$

**Case 11:** We take the following formulae for  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (57)$$

For dark soliton solution, we take  $R(x) = R_0 \tanh(x)$

$$u(x, z) = R_0 \tanh(x) e^{\Upsilon}, \quad (58)$$

and the potential is given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) - \left[ \frac{1}{2} n(n+1) + R_0^2 + R_0^2 U_2 \right] \tanh^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \sinh^3(x), \quad (59)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \coth^{2n}(x) \tanh^3(x), \quad (60)$$

where

$$g(x) = -2\beta_2 R_0^2 \coth^{2n}(x) \ln(\cosh(x)) - 2\beta_2 R_0^2 \coth^{2n}(x) \operatorname{sech}^2(x) + \frac{1}{2}\beta_2 R_0^2 \coth^{2n}(x) \operatorname{sech}^4(x) + \frac{3}{2}\beta_1 \coth^{2n}(x) \cosh(x) - \frac{1}{6}\beta_1 \coth^{2n}(x) \cosh(3x). \quad (61)$$



**Case 12:** We take the following expressions for  $V_1(x)$  and  $V_2(x)$

$$V_1(x) = \beta_1 \tanh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \tanh^{2n}(x) \tanh^3(x). \quad (62)$$

We singular soliton solution we take  $R(x) = R_0 \coth(x)$ :

$$u(x, z) = R_0 \coth(x) e^{\Upsilon}, \quad (63)$$

and the associated potentials are given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) - \left[ \frac{1}{2} n(n+1) + R_0^2 + R_0^2 U_2 \right] \coth^2(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\coth^2(x)} \right) + i\beta_1 \tanh^{2n}(x) \sinh^3(x), \quad (64)$$

$$D_{nl}(x) = U_2(x) + i\beta_2 \tanh^{2n}(x) \tanh^3(x), \quad (65)$$

where

$$g(x) = -2\beta_2 R_0^2 \tanh^{2n}(x) \ln(\cosh(x)) + \frac{3}{2} \beta_1 \tanh^{2n}(x) \cosh(x) - \frac{1}{6} \beta_1 \tanh^{2n}(x) \cosh(3x). \quad (66)$$

## 2.2. PARABOLIC-LAW NONLINEARITY

In parabolic-law nonlinear media, we have  $F(R^2) = R^2 + \eta R^4$ . Now Eq. (3) takes the form

$$\frac{1}{2} n R \frac{d^2 R}{dx^2} + \frac{1}{2} n(n-1) \left( \frac{dR}{dx} \right)^2 - \left[ \frac{1}{2} n^2 g^2(x) + n\rho - U_1(x) \right] R^2 + U_2(x) R^4 + R^4 + \eta R^6 = 0. \quad (67)$$

Here we use similar analytic forms of potentials as in the case of Kerr law media.

**Case 1:** We take the following forms of potentials  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (68)$$

The corresponding linear complex potential for bright soliton solution is given by

$$D_l(x) = \left( \frac{1}{2} n^2 g^2(x) + \left[ \frac{1}{2} n(n+1) - R_0^2 - R_0^2 U_2 \right] \operatorname{sech}^2(x) + \eta R_0^4 \operatorname{sech}^4(x) + n\rho - \frac{1}{2} n^2 \right) + i\beta_1 \cosh^{2n}(x) \sinh^3(x). \quad (69)$$

**Case 2:** We assign the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (70)$$

For dark soliton solution, the complex-valued potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + \eta R_0^4 \tanh^4(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \sinh^3(x). \quad (71)$$

**Case 3:** We assume the following expressions of the potentials  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (72)$$

Thus for the bright soliton solution, the external potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2U_2 \right] \operatorname{sech}^2(x) + \eta R_0^4 \operatorname{sech}^4(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \tanh^3(x). \quad (73)$$

**Case 4:** We take the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (74)$$

For dark soliton solution, the linear complex-valued potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + \eta R_0^4 \tanh^4(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \tanh^3(x). \quad (75)$$

**Case 5:** We take the following forms of the potentials  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (76)$$

The corresponding complex-valued potential for bright soliton solution is given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2U_2 \right] \operatorname{sech}^2(x) + \eta R_0^4 \operatorname{sech}^4(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \tanh^3(x). \quad (77)$$

**Case 6:** We take the following expressions of the potentials  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (78)$$

The linear complex-valued potential corresponding to dark soliton solution is obtained

as follows

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + \eta R_0^4 \tanh^4(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \tanh^3(x). \quad (79)$$

**Case 7:** We choose the following forms of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (80)$$

Then the complex external potential for bright soliton solution is given by:

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 - R_0^2U_2 \right] \operatorname{sech}^2(x) + \eta R_0^4 \operatorname{sech}^4(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \sinh^3(x). \quad (81)$$

**Case 8:** We take the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (82)$$

The corresponding potentials for dark soliton solution are as follows

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) + R_0^2 + R_0^2U_2 \right] \tanh^2(x) + \eta R_0^4 \tanh^4(x) + n\rho + n^2 - \frac{1}{2} \frac{n(n-1)}{\tanh^2(x)} \right) + i\beta_1 \coth^{2n}(x) \sinh^3(x). \quad (83)$$

### 2.3. POWER-LAW NONLINEARITY

For optical media with power-law nonlinearities, we take  $F(R^2) = R^{2r}$ , where  $r > 1$  is the order of nonlinearity. Thus Eq. (3) takes the form

$$\frac{1}{2}nR \frac{d^2R}{dx^2} + \frac{1}{2}n(n-1) \left( \frac{dR}{dx} \right)^2 - \left[ \frac{1}{2}n^2g^2(x) + n\rho - U_1(x) \right] R^2 + U_2(x)R^4 + R^{2r+2} = 0. \quad (84)$$

**Case 1:** We consider the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (85)$$

For exact bright soliton solution, the potentials are obtained as follows

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2U_2 \right] \operatorname{sech}^2(x) - R_0^{2r} \operatorname{sech}^{2r}(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \sinh^3(x). \quad (86)$$

**Case 2:** We take the following expressions of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (87)$$

The complex-valued potential for dark soliton solution is given by

$$D_l(x) = \left( \frac{1}{2}n^2 g^2(x) - \left[ \frac{1}{2}n(n+1) - R_0^2 U_2 \right] \tanh^2(x) - R_0^{2r} \tanh^{2r}(x) + n\rho + n^2 + i\beta_1 \coth^{2n}(x) \sinh^3(x) \right). \quad (88)$$

**Case 3:** We use the following explicit expressions for  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (89)$$

Then for bright soliton solution, the external complex-valued potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2 g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 U_2 \right] \operatorname{sech}^2(x) - R_0^{2r} \operatorname{sech}^{2r}(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \tanh^3(x). \quad (90)$$

**Case 4:** We assume the following forms of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (91)$$

For dark soliton solution, we get the potential

$$D_l(x) = \left( \frac{1}{2}n^2 g^2(x) - \left[ \frac{1}{2}n(n+1) - R_0^2 U_2 \right] \tanh^2(x) - R_0^{2r} \tanh^{2r}(x) + n\rho + n^2 \right) + i\beta_1 \coth^{2n}(x) \tanh^3(x). \quad (92)$$

**Case 5:** We choose the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \sinh^3(x). \quad (93)$$

Then for bright soliton solution, the external complex potential is given by

$$D_l(x) = \left( \frac{1}{2}n^2 g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2 U_2 \right] \operatorname{sech}^2(x) - R_0^{2r} \operatorname{sech}^{2r}(x) + n\rho - \frac{1}{2}n^2 \right) + i\beta_1 \cosh^{2n}(x) \tanh^3(x). \quad (94)$$

**Case 6:** We assign the following  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \tanh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \sinh^3(x). \quad (95)$$

The linear complex potential for dark soliton is obtained as follows:

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) - R_0^2U_2 \right] \tanh^2(x) - R_0^{2r} \tanh^{2r}(x) + n\rho + n^2 \right) + i\beta_1 \coth^{2n}(x) \tanh^3(x). \quad (96)$$

**Case 7:** We take the following forms of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \cosh^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \cosh^{2n}(x) \tanh^3(x). \quad (97)$$

Then for bright soliton solution, we get the following exact expression of the potential:

$$D_l(x) = \frac{1}{2}n^2g^2(x) + \left[ \frac{1}{2}n(n+1) - R_0^2U_2 \right] \operatorname{sech}^2(x) - R_0^{2r} \operatorname{sech}^{2r}(x) + n\rho - \frac{1}{2}n^2 + i\beta_1 \cosh^{2n}(x) \sinh^3(x). \quad (98)$$

**Case 8:** We use the following expressions of  $V_1(x)$  and  $V_2(x)$ :

$$V_1(x) = \beta_1 \coth^{2n}(x) \sinh^3(x), \quad V_2(x) = \beta_2 \coth^{2n}(x) \tanh^3(x). \quad (99)$$

The associated external potential for dark soliton solution is then given by

$$D_l(x) = \left( \frac{1}{2}n^2g^2(x) - \left[ \frac{1}{2}n(n+1) - R_0^2U_2 \right] \tanh^2(x) - R_0^{2r} \tanh^{2r}(x) + n\rho + n^2 \right) + i\beta_1 \coth^{2n}(x) \sinh^3(x). \quad (100)$$

### 3. CONCLUSIONS

In this paper, an integer order nonlinear Schrödinger equation, which describes the propagation of optical lattice solitons in nonlinear media with PT-symmetric potentials, has been studied analytically by the inverse engineering method. Three generic forms of optical nonlinearities, namely Kerr law, parabolic law, and power law, have been considered. The following main results have been obtained: i) the Kerr-law nonlinear media supports bright, dark, and singular solitons and ii) the singular solitons do not exist in the parabolic- and power-law media. The exact soliton solutions and their corresponding PT-symmetric external potentials are presented. The obtained results might be useful in the theoretical and experimental studies of PT-symmetric lattice solitons in diverse optics and photonics settings.

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