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EFFECTIVE EVOLUTION OF QUANTUM SYSTEMS USING COARSE GRAINING MAPS

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Abstract. Very often in an experiment the access to a measuring apparatus is very limited. We investigate a possible effective description of a quantum system, when an experimenter has access to an experimental apparatus described by a POVM, and has no information about the quantum system under study. A coarse graining CPTP map is introduced, that maps the state of the quantum system into a compressed effective state, from which one obtains the same measurement statistics.

Key words: Coarse graining, quantum maps, POVM.

1. INTRODUCTION

Description at different scales are fundamental in the study of physical systems. Different notions of coarse graining have been around since the beginning of modern physics [1, 2]. To describe the macroscopic world we use effective theories with great success, the best example being thermodynamics [3]. Recently there has been a lot of interest in defining a coherent view on the concept of quantum coarse graining using the language and tools of quantum information theory developed in the last decades [4–8].

In this paper we define a quantum channel [9, 10], that is a completely positive and trace preserving (CPTP) map, which gives an effective description of a system that is observed by an experimenter through the lens of a positive operator valued measure (POVM) with no other information or access to the system.

The structure of the paper is the following. In subsection 1.1 we give a brief review of the main approaches to quantum coarse graining. In Sec. 2 we remind the necessary mathematical tools and concepts from quantum information theory. In Sec. 3 we propose and characterise a quantum coarse graining map and prove that it is a valid quantum map, namely a CPTP map (completion of proof is given in the Appendix). Then in Sec. 4 we present an application, which describes the general qubit evolutions as coarse grained effective evolutions of a system of a larger dimension. Finally, in Sec. 5 we give a summary and conclusions.

1.1. REVIEW

In the following we describe the main recent approaches in defining a coherent view on the quantum coarse graining concept.

Using the open system representation, the authors of Ref. [4] define a quantum coarse graining as any quantum channel that reduces the dimension of the input Hilbert space and, consequently, the roles of the system and its environment are not so clearly distinguished.

In Ref. [5] one considers that any observer chooses his fundamental unit of information and assigns it a quantum state. Thus different observers can assign states from different state spaces to the same system. Yet, the author shows that this is completely compatible with quantum mechanics and develops a coarse graining procedure that allows to connect the statements of two different observers in complete agreement with the rules of quantum mechanics.

In Ref. [6] the quantum coarse graining is defined operationally in analogy with the classical concept by using bipartitions, and the question of what information can be safely discarded to reduce the complexity of the dynamics is answered. It is also shown how a large class of symmetries can play a role in reducing the complexity of dynamics.

In Ref. [7] the quantum coarse graining of a Hilbert space is obtained by choosing a restricted set of quantum states and applying the method of principal component analysis.

In Ref. [8], using geometrical properties of the discrete phase space of N qubits, a coarse discrete Wigner function is constructed, which represents an effective resultant system.

The first three presented frameworks are mathematically equivalent. Our work belongs to the same category, and the relation between our approach and the first three mentioned above will be explored in a future work.

2. TOOLS

2.1. POSITIVE OPERATOR VALUED MEASURES

The most general form of measurement in quantum mechanics is represented by a positive operator valued measure (POVM). A POVM is a set of positive operators E_i in a Hilbert space H_D that fulfill $\sum_{i=1}^{n} E_i = \mathbb{I}_D$, with D the dimension of H_D and n the number of positive operators E_i (number of outcomes). POVM elements can always be decomposed as $E_i = M_i^\dagger M_i$ in infinitely many ways, thus different sets of M_i can give rise to the same POVM.

The measurement operators are not, in general, orthogonal or commutative, in contrast to projectors. Another difference between a POVM and a projection valued (PV) measure is that the number of outcomes can be different from the dimension of the Hilbert space. In addition a POVM allows the possibility of measurement outcomes associated with nonorthogonal states.

There are many situations in quantum computation and quantum information that require the use of POVM measurements as the optimal way to distinguish a set of quantum states, in which the solution uses POVMs in place of projective measurements [11]. Another aspect is the repeatability of measurements, the classical example being a photon detected by a screen which destroys the photon. For such a measurement, POVMs have to be employed when one can study the measurement statistics, without knowing the post-measurement state. We also mention the Naimark dilation theorem which proves that POVMs can be obtained from PVs acting on a larger Hilbert space [10].

2.2. COMPLETELY POSITIVE TRACE PRESERVING LINEAR MAPS

The most general form of evolution in quantum mechanics is a completely positive and trace preserving linear map, the unitary evolution being a particular case. The trace preserving requirement ensures that the probabilities are conserved, and the complete positivity condition ensures that the map maps quantum states into quantum states [9, 10]. CPTP maps are characterised by the following:

Theorem (Kraus [12]): A linear map $\mathcal{E} : \mathcal{L}(H_D) \longrightarrow \mathcal{L}(H_d)$, where $\mathcal{L}(H_D)$ is the set of linear operators acting on H_D, is completely positive and trace preserving if and only if it can be decomposed into a finite set of n linear operators K_i, with $K_i : H_D \longrightarrow H_d$, named Kraus operators, that satisfy:

$$\forall \psi \in \mathcal{L}(H_D), \text{ one has } \mathcal{E}(\psi) = \sum_{i=1}^{n} K_i \psi K_i^\dagger, \text{with } \sum_{i=1}^{n} K_i^\dagger K_i = \mathbb{I}_D.$$

The number of Kraus operators is not restricted, but it is always possible to define a CPTP map with at most $D \times d$ elements [9, 10].

3. COARSE GRAINING MAP

A coarse graining operation is characterised by a CPTP linear map that reduces the dimension of the state space: $\mathcal{E}_{CG} : \mathcal{L}(H_D) \longrightarrow \mathcal{L}(H_d)$ with $D > d$ [4]. Such a map decreases the number of the degrees of freedom and gives an effective state of a system, when it is impossible or unwanted to describe it in its full detail. Given an arbitrary time dependent unitary evolution Ψ_t of a system of dimension D and a set of d POVM elements that model a saturated detector, with $D > d$, we characterise the quantum map and the d-dimensional effective state that determines all possible

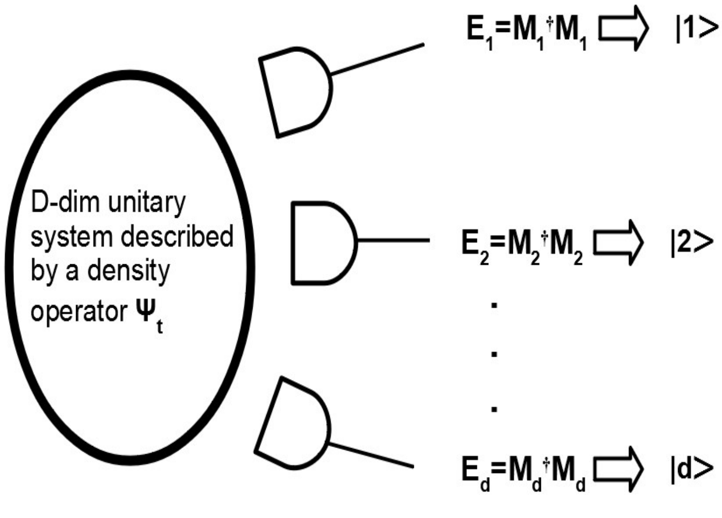

Fig. 1 – Intuitive picture illustrating the coarse graining procedure.

measurement statistics of any experiment done, with the given POVM set.

An intuitive picture is given in Fig. 1. For every POVM element (click of the detector) we associate an element of the new computational basis in the new effective Hilbert space H_d. This means that the diagonal elements of the new coarse grained state are just the probabilities of the outcomes $E_i = M_i^\dagger M_i$, which are given by $\mathrm{Tr}_D\big[E_i\Psi_t\big]$. For the coherence-cross terms, it follows that we sum up all the elements of $M_i\Psi_t M_j^\dagger$. We give next an example of a saturated detector modelled in this way, similar to the one in Ref. [4].

Consider a three level system (qutrit) and a detector that cannot distinguish between the ground state $|1\rangle$ and the first excited state $|2\rangle$ (first two levels), but only between the third level $|3\rangle$ and the first two $|1\rangle$, $|2\rangle$. Since the detector cannot distinguish between $|1\rangle$ and $|2\rangle$, it follows that there can be no coherences in the subspace created by the grouping $|1\rangle$, $|2\rangle$.
It follows that the map is described by:

$\mathcal{E}_{CG}(|1\rangle\langle 1|) = |\tilde{1}\rangle\langle\tilde{1}|$ first click $\mathcal{E}_{CG}(|1\rangle\langle 2|) = 0$ no coherence
$\mathcal{E}_{CG}(|2\rangle\langle 2|) = |\tilde{1}\rangle\langle\tilde{1}|$ first click $\mathcal{E}_{CG}(|2\rangle\langle 1|) = 0$ no coherence
$\mathcal{E}_{CG}(|3\rangle\langle 3|) = |\tilde{2}\rangle\langle\tilde{2}|$ second click $\mathcal{E}_{CG}(|1\rangle\langle 3|) = c|\tilde{1}\rangle\langle\tilde{2}|$ coherence
$\mathcal{E}_{CG}(|2\rangle\langle 3|) = c|\tilde{1}\rangle\langle\tilde{2}|$ coherence $\mathcal{E}_{CG}(|3\rangle\langle 1|) = c|\tilde{2}\rangle\langle\tilde{1}|$ coherence
$\mathcal{E}_{CG}(|3\rangle\langle 2|) = c|\tilde{2}\rangle\langle\tilde{1}|$ coherence

We see the transition from the Hilbert space H_D of the system to the effective Hilbert space H_d, where operators $|i\rangle\langle j| \in H_D$ and $|\tilde{x}\rangle\langle\tilde{y}| \in H_d$. The role of the constant c will be clarified below. Then, given the POVM $E_1 = |1\rangle\langle 1| + |2\rangle\langle 2|, M_1 = |1\rangle\langle 1| + |2\rangle\langle 2|, E_2 = |3\rangle\langle 3|, M_2 = |3\rangle\langle 3|$ and a three dimensional state $\psi \in \mathcal{L}(H_3)$, the two-dimensional coarse grained state $\rho = \mathcal{E}_{CG}(\psi) \in \mathcal{L}(H_2)$ will be:

$$\rho = \mathcal{E}_{CG}(\psi) = \begin{pmatrix} \psi_{11} + \psi_{22} & c\,(\psi_{13} + \psi_{23}) \\ c\,(\psi_{31} + \psi_{32}) & \psi_{33} \end{pmatrix}.$$

Here $\psi_{11} + \psi_{22}$ and ψ_{33} are the probabilities of outcome E_1 and, respectively E_2, the non-diagonal elements correspond to the coherence-cross terms, and the constant c is chosen such that $\mathcal{E}_{CG}$ is a CPTP linear map.

Given the POVM set $E_i = M_i^\dagger M_i$ of d elements with $\sum\limits_{i=1}^{d} E_i = \mathbb{I}_D$ and a unitary evolution of a D-dimensional system, described by the state Ψ_t, and based on the described coarse graining procedure, it follows that the map $\mathcal{E}_{CG} : \mathcal{L}(H_D) \to \mathcal{L}(H_d)$ has to be taken of the form

$$\mathcal{E}_{CG}(\Psi_t) = \sum_{k,l=1}^{d} \mathrm{Tr}_D \left[M_k \Psi_t M_l^\dagger T_{k,l} \right] |k\rangle_d \langle l|_d. \tag{1}$$

Since we want to obtain the map described previously, it results that $T_{k,l}$ has to be given by the expression

$$T_{k,l} = cA + (\mathbb{I}_D - cA)\delta_{k,l},$$

where c is a constant and $A = \sum\limits_{i,j=1}^{D} |i\rangle\langle j|$. A has the property that $\mathrm{Tr}\left[XA\right] = \sum_{i,j} x_{ij}$, where x_{ij} are the matrix elements of the operator X. It is easy to see that on the diagonal ($k = l$) $T_{k,l} = \mathbb{I}_D$ and for the off-diagonal elements ($k \neq l$) we get the operator $M_k \Psi_t M_l^\dagger cA$ of which we take the trace and the result is the sum of all elements of $cM_k \Psi_t M_l^\dagger$, as required. Then the coarse grained state is

$$\rho_t^d = \mathcal{E}_{CG}(\Psi_t).$$

We still have to show that the map is CPTP, and for this purpose we will use the Kraus theorem [9, 10, 12]. Therefore, we will construct a Kraus form of the map, which was presented above in a tomographic form. We start with the following ansatz for the Kraus operators:

$$K_{i,j} = \sum_{k=1}^{d} |k\rangle_d \langle j|_D X_{i,k} M_k, \text{of dimensions } D \times d, \text{ where } \quad i = 1,...,d, \quad j = 1,...,D.$$

We notice that their number is $D \times d$, the maximal number guaranteed by the Choi theorem [10, 13]. We can easily verify that:

$$\mathcal{E}_{CG}(\Psi_t) = \sum_{i=1}^{d}\sum_{j=1}^{D} K_{i,j}\Psi_t K_{i,j}^{\dagger} = \sum_{i=1}^{d}\sum_{j=1}^{D}\sum_{k,l=1}^{d} \langle j|_D X_{i,k} M_k \Psi_t M_l^{\dagger} X_{i,l}^{\dagger}|j\rangle_D |k\rangle_d\langle l|_d.$$

By using the independence of the operators on j, linearity and the cyclic permutation of the trace, we obtain:

$$\mathcal{E}_{CG}(\Psi_t) = \sum_{k,l=1}^{d} \mathrm{Tr}_D\left[M_k\Psi_t M_l^{\dagger}\sum_{i=1}^{d} X_{l,i}^{\dagger}X_{i,k}\right]|k\rangle_d\langle l|_d. \qquad (2)$$

Comparing the two forms (1) and (2), we require that:

$$\sum_{i=1}^{d} X_{l,i}^{\dagger}X_{i,k} = cA + (\mathbb{I}_D - cA)\delta_{k,l},$$

for which we find the solution (Appendix):

$$X_{i,k} = \frac{1}{D\sqrt{d}}A + (\mathbb{I}_D - \frac{1}{D}A)\delta_{i,k}.$$

It follows that:

$$\sum_{i=1}^{d} X_{l,i}^{\dagger}X_{i,k} = \frac{1}{D}A + (\mathbb{I}_D - \frac{1}{D}A)\delta_{k,l}, \qquad (3)$$

where we used the fact that the largest value of c for which the considered map remains CPTP and the coarse graining is maximally coherent, is $\frac{1}{D}$. The proof is given in the Appendix. By knowing the form of the operators $X_{i,k}$, we can now prove that the map is CPTP. Indeed, we have $K_{i,j} = \sum_{k=1}^{d} |k\rangle_d\langle j|_D X_{i,k}M_k$, where $i = 1,...,d$, $j = 1,...,D$. Then

$$K_{i,j}^{\dagger}K_{i,j} = \sum_{k,l=1}^{d} M_l^{\dagger}X_{i,l}^{\dagger}|j\rangle_D\langle l|_d|k\rangle_d\langle j|_D X_{i,k}M_k = \sum_{k=1}^{d} M_k^{\dagger}X_{i,k}^{\dagger}|j\rangle_D\langle j|_D X_{i,k}M_k.$$

By summing over j (identity), we get:

$$\sum_{j=1}^{D} K_{i,j}^{\dagger}K_{i,j} = \sum_{k=1}^{d} M_k^{\dagger}X_{i,k}^{\dagger}X_{i,k}M_k.$$

We sum over i, and using (3) we obtain:

$$\sum_{i=1}^{d}\sum_{j=1}^{D} K_{i,j}^{\dagger}K_{i,j} = \sum_{k=1}^{d} M_k^{\dagger}\left[\frac{1}{D}A + \delta_{k,k}(\mathbb{I}_D - \frac{1}{D}A)\right]M_k = \sum_{k=1}^{d} M_k^{\dagger}M_k.$$

By definition the POVM elements sum up to identity:

$$\sum_{i=1}^{d}\sum_{j=1}^{D} K_{i,j}^{\dagger}K_{i,j} = \sum_{k=1}^{d} E_k = \mathbb{I}_D,$$

and, therefore, we can write the map as required:

$$\rho_t^d = \sum_{i=1}^{d}\sum_{j=1}^{D} K_{i,j}\Psi_t K_{i,j}^{\dagger}.$$

4. QUBIT EVOLUTIONS

We will show that, by using the constructed map, almost all qubit quantum evolutions can be seen as coarsed grained by a saturated detector from a qutrit ($D = 3$). For an arbitrary qubit quantum evolution we give the elements of the map needed to obtain it from a qutrit. The construction is not unique, in fact there is an infinite number of them, similar to the open system representation. In the following we choose one of them, based on its simplicity.

The general evolution of a qubit is given by:

$$\rho_t = \begin{pmatrix} p(t) & r(t)e^{\mathrm{i}\varphi(t)} \\ r(t)e^{-\mathrm{i}\varphi(t)} & 1-p(t) \end{pmatrix},$$

with $0 \leq r(t) \leq \sqrt{p(t)(1-p(t))}$ being the Bloch sphere condition and $p(t)$ a probability function.

We take the pure state of the qutrit of the form:

$$|\Psi_t\rangle = \begin{pmatrix} \sqrt{p(t)}\cos[x(t)] \\ \sqrt{p(t)}\sin[x(t)] \\ \sqrt{1-p(t)}e^{-\mathrm{i}\varphi(t)} \end{pmatrix},$$

where $x(t)$ is an arbitrary function of time.

The POVM set and its chosen decomposition are:

$$E_1 = |1\rangle\langle 1| + |2\rangle\langle 2|,\ M_1 = \frac{1}{\sqrt{2}}(|1\rangle\langle 1| + |3\rangle\langle 1|) + |2\rangle\langle 2|,$$
$$E_2 = |3\rangle\langle 3|,\ M_2 = \frac{1}{\sqrt{3}}(|1\rangle\langle 3| + |2\rangle\langle 3| + |3\rangle\langle 3|).$$

We have (for simplicity we omit for the moment to write the argument t):

$$|\Psi_t\rangle\langle\Psi_t| = \begin{pmatrix} p\cos^2[x] & p\cos[x]\sin[x] & \sqrt{p(1-p)}\cos[x]e^{\mathrm{i}\varphi} \\ p\cos[x]\sin[x] & p\sin^2[x] & \sqrt{p(1-p)}\sin[x]e^{\mathrm{i}\varphi} \\ \sqrt{p(1-p)}\cos[x]e^{-\mathrm{i}\varphi} & \sqrt{p(1-p)}\sin[x]e^{-\mathrm{i}\varphi} & 1-p \end{pmatrix}.$$

The constructed map, applied to an arbitrary qutrit state ρ is:

$$\mathcal{E}_{CG}(\rho) = \begin{pmatrix} \rho_{11}+\rho_{22} & \frac{\sqrt{2}\rho_{13}+\rho_{23}}{\sqrt{3}} \\ \frac{\sqrt{2}\rho_{31}+\rho_{32}}{\sqrt{3}} & \rho_{33} \end{pmatrix}.$$

Now applying the map to the state $|\Psi_t\rangle\langle\Psi_t|$, we obtain:

$$\rho_t = \mathcal{E}_{CG}(|\Psi_t\rangle\langle\Psi_t|) = \begin{pmatrix} p & \sqrt{p(1-p)}e^{\mathrm{i}\varphi}\frac{\sqrt{2}\cos[x]+\sin[x]}{\sqrt{3}} \\ \sqrt{p(1-p)}e^{-\mathrm{i}\varphi}\frac{\sqrt{2}\cos[x]+\sin[x]}{\sqrt{3}} & 1-p \end{pmatrix}.$$

Using the identity $\frac{\sqrt{2}\cos[x]+\sin[x]}{\sqrt{3}} = \sin[x+\tan^{-1}(\sqrt{2})]$ and $x(t) = \arcsin[f(t)] - \tan^{-1}(\sqrt{2})$, with $0 \le f(t) \le 1$ (arbitrary function), we get the coarse grained state:

$$\rho_t = \begin{pmatrix} p(t) & f(t)\sqrt{p(t)(1-p(t))}e^{\mathrm{i}\varphi(t)} \\ f(t)\sqrt{p(t)(1-p(t))}e^{-\mathrm{i}\varphi(t)} & 1-p(t) \end{pmatrix}.$$

We now split the evolutions into the following classes.
a) For $f(t)=1$ we obtain a general unitary evolution of the qubit:

$$\rho_t^a = \begin{pmatrix} p(t) & \sqrt{p(t)(1-p(t))}e^{\mathrm{i}\varphi(t)} \\ \sqrt{p(t)(1-p(t))}e^{-\mathrm{i}\varphi(t)} & 1-p(t) \end{pmatrix}.$$

b) For $f(t)=0$ we obtain the diagonal evolution $\rho_t^b = \begin{pmatrix} p(t) & 0 \\ 0 & 1-p(t) \end{pmatrix}$.
c) For $f(t) = \frac{r(t)}{\sqrt{p(t)(1-p(t))}}$ we get the general evolution:

$$\rho_t^c = \begin{pmatrix} p(t) & r(t)e^{\mathrm{i}\varphi(t)} \\ r(t)e^{-\mathrm{i}\varphi(t)} & 1-p(t) \end{pmatrix},$$

with the limitation that we can go only arbitrarily close to the states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, in order to avoid an indeterminate form. Thus we obtained almost all qubit evolutions. The question of finding similar construction of larger dimensions, *i.e.* qudits, will be investigated in a future work. It can be viewed as an analogous way of defining the cut between an open system and its environment [14, 15].

5. SUMMARY AND CONCLUSIONS

We constructed a new quantum map that gives an effective description of a system observed through a coarse grained measurement (POVM) by an experimenter with no access or additional information about the system. We suppose that the proposed map could have a direct relationship with the frameworks in Refs. [4, 6], that remains to be explored. Though not so general, our framework seems to be more simple for analysing the link between coarse grained effective descriptions of unitary systems and memory effects (non-Markovianity), which is another subject of interest. Another avenue to be explored is to further develop an alternative way of defining the 'cut' between a system and its 'environment' for $d \geq 3$ (having done so for $d = 2$).

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A. APPENDIX

We need to show that:

$$X_{i,k} = \frac{1}{D\sqrt{d}}A + (\mathbb{I}_D - \frac{1}{D}A)\delta_{i,k}$$

and that $\frac{1}{D}$ is the maximum value of the constant c for which one has:

$$\sum_{i=1}^{d} X_{l,i}^{\dagger} X_{i,k} = \frac{1}{D} A + (\mathbb{I}_D - \frac{1}{D} A)\delta_{k,l}.$$

Knowing that for $A = \sum_{i,j=1}^{D} |i\rangle\langle j|$ we have $A^2 = DA$ we start with the obvious ansatz:

$$X_{i,k} = c_1 A + (\mathbb{I}_D - c_2 A)\delta_{i,k},$$

with $c_1, c_2 \in \mathbb{C}$. We easily see that:

$$X_{l,i}^{\dagger} X_{i,k} = \mid c_1 \mid^2 DA + \big(c_1^* - c_1^* c_2 D\big) A\delta_{i,k}$$
$$+ \big(c_1 - c_2^* c_1 D\big) A\delta_{l,i} + \big(\mathbb{I}_D - c_2 A - c_2^* A + \mid c_2 \mid^2 DA\big)\delta_{l,i}\delta_{i,k}.$$

We sum over i using the $\delta_{i,j}$ properties and get:

$$\sum_{i=1}^{d} X_{l,i}^{\dagger} X_{i,k} = \mid c_1 \mid^2 DdA + \big(c_1^* - c_1^* c_2 D\big) A$$
$$+ \big(c_1 - c_2^* c_1 D\big) A + \big\{\mathbb{I}_D - \big(c_2 + c_2^* - \mid c_2 \mid^2\big) A\big\}\delta_{k,l}$$
$$= \big\{ \mid c_1 \mid^2 Dd + 2\mathrm{Re}(c_1) - 2D\mathrm{Re}(c_1 \times c_2)\big\} A + \big\{\mathbb{I}_D - \big(2\mathrm{Re}(c_2) - \mid c_2 \mid^2 D\big) A\big\}\delta_{k,l}.$$

Now we have an optimization with constraint problem, namely:

$$\textit{Maximize}\{\mid c_1 \mid^2 Dd + 2\mathrm{Re}(c_1) - 2D\mathrm{Re}(c_1 \times c_2)\},$$
$$\textit{subject to} \mid c_1 \mid^2 Dd + 2\mathrm{Re}(c_1) - 2D\mathrm{Re}((c_1 \times c_2)) = 2\mathrm{Re}(c_2) - \mid c_2 \mid^2 D.$$

We apply the Lagrange multipliers method and for the simplicity of the proof we take c_1, c_2 real (for complex c_1, c_2 the solution is the same):

$$f(c_1, c_2) = c_1^2 Dd + 2c_1 - 2Dc_1 c_2,$$
$$g(c_1, c_2) = f(c_1, c_2) - 2c_2 + c_2^2 D,$$
$$L(c_1, c_2, \lambda) = f(c_1, c_2) - \lambda g(c_1, c_2).$$

From $\nabla L(c_1, c_2, \lambda) = \vec{0}$ we get the system:

$$2(1-\lambda)(Ddc_1 - Dc_2 + 1) = 0,$$
$$-2Dc_1(1-\lambda) + 2\lambda - 2D\lambda c_2 = 0,$$
$$-Dc_2^2 + 2c_2 - c_1^2 Dd - 2c_1 + 2Dc_1 c_2 = 0,$$

with the solutions: $\lambda = 1, \frac{1}{1-d}$ and $c_1 = \pm\frac{1}{D\sqrt{d}}, \pm\frac{1}{D\sqrt{d(d-1)}}$, $c_2 = \frac{1}{D}, \frac{1\pm\sqrt{\frac{d}{d-1}}}{D}$.
It can easily be seen that $f(c_1 = \pm\frac{1}{D\sqrt{d}}, c_2 = \frac{1}{D}) = \frac{1}{D}$ is the maximum.