Romanian Reports in Physics **72**, 417 (2020)

OPTICAL SOLITONS IN NON-KERR NONLINEAR MEDIA WITH AN IMPRINTED PARITY-TIME-SYMMETRIC MIXED LINEAR-NONLINEAR LATTICE

SYED TAHIR RAZA RIZVI[1], IJAZ ALI[1], SYED OAN ABBAS[1], KASHIF ALI[1], QIN ZHOU[2,*]

[1]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan
[2]School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan 430212, China
[*]Corresponding author, Email: qinzhou@whu.edu.cn

Received August 10, 2020

Abstract. We study families of optical solitons that form in three different types of nonlinear media with parity-time-symmetric mixed linear-nonlinear lattice potentials. The considered optical nonlinearities have the forms of Kerr law, quadratic-cubic law, and anti-cubic law. We obtain various types of exact soliton solutions with the help of the inverse engineering technique. Under diverse physical conditions, we give the explicit expressions for dark, bright, and singular soliton solutions.

Key words: Parity-time symmetry, optical lattice, inverse engineering technique, complex-valued potential.

1. INTRODUCTION

In 1998, Boettcher and Bender [1] proposed the idea of parity-time symmetric complex-valued potentials in quantum mechanics. A complex-valued potential $V(x)$ of a non-Hermitian Hamiltonian is parity-time (PT) symmetric when its real part is an even function and the associated imaginary part is an odd function [2–17]. By introducing a complex-valued refractive index profile $p(x) = p_R(x) + ip_I(x)$, the PT-symmetric complex potential can be achieved in optical lattices, in which the real part of the refractive index is an even function $p_R(x) = p_R(-x)$ and the imaginary part of the refractive index is an odd function $p_I(x) = -p_I(-x)$. There are many applications of PT symmetry in various areas such as electronics [18, 19], quantum field theory [20], microwave cavities [21], and in linear [22] and nonlinear [23, 24] optics. New types of soliton structures such as multihump and vortex solitons can be formed in PT-symmetric external potentials [25–27]. Optical solitons can form and can propagate stably in different kinds of PT-symmetric lattices. Such solitons are self-localized wave packets that can be created when the combined actions of the diffraction/dispersion effect, the PT-symmetric modulation of the refractive index, and the effect of optical nonlinearity are exactly balanced. The inverse engineering technique is the basic tool to obtain exact soliton solutions in PT-symmetric optical lattices [28, 29]. Recently, the existence, stability, and dynamics of solitons in media with imprinted PT-symmetric mixed linear and nonlinear optical lattices have been studied by different research groups [30–33]. In this work, we study optical solitons in PT-symmetric mixed linear-nonlinear lattices with Kerr law, quadratic-cubic law, and anti-cubic law nonlinearities. As a result, the corresponding PT-symmetric mixed linear and nonlinear complex-valued potentials along with the exact forms of such optical solitons, are reported.

The propagation of nonlinear waves in nonlinear media with an imprinted PT-symmetric mixed linear-nonlinear optical lattices is described by the complex nonlinear Schrödinger equation (CNLSE) [34]:

$$iq_z + \frac{1}{2}q_{xx} + U_l(x)q + U_{nl}(x)|q|^2 q + F(|q|^2)q = 0, \tag{1}$$

where the complex field amplitude is represented by $q(x,z)$, the normalized transverse coordinate is x, and z is the longitudinal coordinate. Here the complex-valued linear potential is $U_l(x) = V_1(x) + iW_1(x)$ and the complex-valued nonlinear potential is $U_{nl}(x) = V_2(x) + iW_2(x)$. It should be noted that the PT symmetry imposes two constraints on the complex potentials: the real parts of complex potentials must be even functions and the imaginary parts of complex potentials must be odd functions. The last term in Eq. (1) stands for the non-Kerr law nonlinearity.

2. THEORETICAL ANALYSIS

With the aid of inverse engineering technique, we obtain exact soliton solutions of Eq. (1). For this purpose, we first choose the following ansatz:

$$q(x,z) = C(x)e^{\Gamma}, \tag{2}$$

where,

$$\Gamma = i(\lambda z + \int g(x)dx). \tag{3}$$

Here $C(x)$ is the real amplitude, while $g(x)$ is the non-homogeneous phase of the mode and λ is the propagation constant. Replacing Eq. (2) into Eq. (1), then we get the differential equations for $C(x)$ and the exact expression of the function $g(x)$:

$$\frac{1}{2}\frac{d^2C}{dx^2} - \left(\frac{1}{2}g^2 + \lambda - V_1(x)\right)C + V_2(x)C^3 + F(C^2)C = 0, \tag{4}$$

and

$$g(x) = -\frac{2}{C^2}\int C^2\left[W_2(x)C^2 + W_1(x)\right]dx. \tag{5}$$

We then follow these two steps:
Step 1. For the known functions $C(x)$, $W_1(x)$, and $W_2(x)$ we get the phase gradient $g(x)$ from Eq. (5). For bright solitons we choose $C(x) = C_0\,\mathrm{sech}(x)$, for dark solitons we select $C(x) = C_0\tanh(x)$, and for singular soliton we consider $C(x) = C_0\coth(x)$. Here, C_0 indicates the soliton amplitude.
Step 2. By replacing $C(x)$ and $g(x)$ into Eq. (4), we get the relation between $V_1(x)$ and $V_2(x)$. We thus obtain $V_1(x)$ if $V_2(x)$ is given or we obtain $V_2(x)$ if $V_1(x)$ is given.

2.1. KERR LAW NONLINEARITY

For Kerr law nonlinearity [35], we consider $F(C) = C$. Thus Eq. (4) reduces to

$$\frac{1}{2}\frac{d^2C}{dx^2} - \left(\frac{1}{2}g^2 + \lambda - V_1(x)\right)C + V_2(x)C^3 + C^3 = 0. \tag{6}$$

2.1.1. The imaginary parts of complex linear and nonlinear potentials having the form of product of sine and cosine hyperbolic functions.

Here we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1\cosh(x)\sinh(x), \tag{7}$$

$$W_2(x) = \omega_2\cosh(x)\sinh(x), \tag{8}$$

In order to get bright solitons we consider $C(x) = C_0\,\mathrm{sech}(x)$, then we have

$$q(x,z) = C_0\,\mathrm{sech}(x)e^{\Gamma}, \tag{9}$$

and the linear and nonlinear potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1\cosh(x)\sinh(x) + \left(1 - C_0^2 - C_0^2V_2(x)\right)\mathrm{sech}^2(x), \tag{10}$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \tag{11}$$

where

$$g(x) = \omega_2(\cosh^2(x)\,\mathrm{sech}^2(x)) - \frac{2\omega_1}{C_0^2}\left(\cosh^2(x)\ln(\cosh(x))\right). \tag{12}$$

For dark solitons we consider $C(x) = C_0\tanh(x)$ and we get

$$q(x,z) = C_0\tanh(x)e^{\Gamma}, \tag{13}$$

and the corresponding linear and nonlinear potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\tanh^2(x), \tag{14}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{15}$$

where

$$g(x) = -2\omega_2\left(\sinh^2(x) - 2\coth^2(x)\ln(\cosh(x)) + 1\right) - \frac{\omega_1}{C_0^2}\left(\cosh^2(x) - 2\coth^2(x)\ln(\cosh(x))\right). \tag{16}$$

For singular solitons we consider $C(x) = C_0 \coth(x)$ and we have

$$q(x,z) = C_0 \coth(x) e^{\Gamma}, \tag{17}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\coth^2(x), \tag{18}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{19}$$

where

$$g(x) = -\omega_2\left(\cosh^2(x) + 4\tanh^2(x)\ln(\sinh(x)) - 1\right) - \frac{\omega_1}{C_0^2}\left(\sinh^2(x) + 2\ln(\sinh(x))\right). \tag{20}$$

2.1.2. The imaginary parts of complex linear and nonlinear potentials having the form of product of secant and sine hyperbolic functions.

Here we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \tag{21}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{22}$$

In order to get bright solitons we consider $C(x) = C_0 \operatorname{sech}(x)$ and we get

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \tag{23}$$

and the interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \operatorname{sech}(x)\sinh(x) + \left(1 - C_0^2 - C_0^2 V_2(x)\right)\operatorname{sech}^2(x), \tag{24}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{25}$$

where

$$g(x) = -2\omega_2\left(\frac{\tanh^2(x)}{4} + \frac{\sinh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\sinh^2(x). \tag{26}$$

For dark solitons we consider $C(x) = C_0 \tanh(x)$ and we have

$$q(x,z) = C_0 \tanh(x) e^{\Gamma}, \tag{27}$$

and the corresponding linear and nonlinear potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\tanh^2(x), \tag{28}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{29}$$

where

$$g(x) = -2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} + \frac{\tanh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2})\right). \tag{30}$$

For singular solitons, we consider $C(x) = C_0 \coth(x)$ and we get

$$q(x,z) = C_0 \coth(x) e^{\Gamma}, \tag{31}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\coth^2(x), \tag{32}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{33}$$

where

$$g(x) = -2\omega_2\left(\ln(\sinh(x))\tanh^2(x) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{34}$$

2.1.3. The imaginary part of linear potential is the product of hyperbolic cosine and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic secant and hyperbolic sine function.

Now we consider the following $W_1(x)$ and $W_2(x)$.

$$W_1(x) = \omega_1 \cosh(x)\sinh(x), \tag{35}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{36}$$

In order to get bright solitons we consider $C(x) = C_0 \operatorname{sech}(x)$ and we get

$$q(x,z) = C_0 \operatorname{sech}(x)e^{\Gamma}, \tag{37}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \cosh(x)\sinh(x) + \left(1 - C_0^2 - C_0^2 V_2(x)\right)\operatorname{sech}^2(x), \tag{38}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{39}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x) + \frac{\sinh^2(x)}{4}\right) + \frac{-2\omega_1}{C_0^2}\cosh^2(x)\ln(\cosh(x)). \tag{40}$$

For dark solitons, we consider $C(x) = C_0 \tanh(x)$ and we get

$$q(x,z) = C_0 \tanh(x)e^{\Gamma}, \tag{41}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\tanh^2(x), \tag{42}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{43}$$

where

$$g(x) = 2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} - \frac{\tanh^2(x)}{4}\right) + \frac{-2\omega_1}{C_0^2}\left(\frac{\cosh^2(x)}{2} - \coth^2(x)\ln(\cosh(x))\right). \tag{44}$$

For singular solitons, we consider $C(x) = C_0 \coth(x)$ and we have

$$q(x,z) = C_0 \coth(x)e^{\Gamma}, \tag{45}$$

and the corresponding linear and nonlinear potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\coth^2(x), \tag{46}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{47}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x)\ln(\sinh(x)) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\frac{\sinh^2(x)}{2} + \tanh^2(x)\ln(\sinh(x))\right). \tag{48}$$

2.1.4. The imaginary part of the linear potential is the product of hyperbolic secant and hyperbolic sine function and the imaginary part of the nonlinear potential is the product of hyperbolic cosine and hyperbolic sine function.

Here we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \tag{49}$$

$$W_2(x) = \omega_2 \cosh(x)\sinh(x), \tag{50}$$

For bright solitons, we consider $A(x) = A_0 \operatorname{sech}(x)$, then

$$q(x,z) = C_0 \operatorname{sech}(x)e^{\Gamma}, \tag{51}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + i\omega_1 \operatorname{sech}(x)\sinh(x) - \frac{1}{2} + \left(1 - C_0^2 - C_0^2 V_2(x)\right)\operatorname{sech}^2(x) +, \tag{52}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{53}$$

where

$$g(x) = -\omega_2 \sinh^2(x) - \frac{\omega_1}{C_0^2}\left(\sinh^2(x)\right). \tag{54}$$

For dark solitons, we consider $C(x) = C_0 \tanh(x)$, then

$$q(x,z) = C_0 \tanh(x)e^{\Gamma}, \tag{55}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\tanh^2(x), \tag{56}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{57}$$

where

$$g(x) = -2\omega_2\left(1 - \ln(\cosh(x))\coth^2(x) + \sinh^2(x)\right) - \frac{2\omega_1}{C_0^2}\left(\cosh(x)\coth^2(x) - \frac{1}{2}\right). \tag{58}$$

For singular solitons, we consider $C(x) = C_0 \coth(x)$, then

$$q(x,z) = C_0 \coth(x) e^{\Gamma}, \tag{59}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + C_0^2 + C_0^2 V_2(x)\right)\coth^2(x), \tag{60}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{61}$$

where

$$g(x) = -2\omega_2\left(\cosh^2(x) + 2\ln(\sinh(x))\tanh^2(x) - 1\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{62}$$

2.2. QUADRATIC-CUBIC LAW

For quadratic-cubic law [35], we consider $F(C) = a_1\sqrt{C} + a_2 C$, where a_1 and a_2 are constants. Thus Eq. (4) reduces to

$$\frac{1}{2}\frac{d^2C}{dx^2} - \left(\frac{1}{2}g^2 + \lambda - V_1(x)\right)C + V_2(x)C^3 + a_1 C^2 + a_2 C^3 = 0, \tag{63}$$

2.2.1. The imaginary parts of complex linear and nonlinear potentials having the form of product of hyperbolic sine and hyperbolic cosine functions.

Now we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \cosh(x)\sinh(x), \tag{64}$$

$$W_2(x) = \omega_2 \cosh(x)\sinh(x), \tag{65}$$

In order to get bright solitons we consider $C(x) = C_0\,\mathrm{sech}(x)$, then we have

$$q(x,z) = C_0\,\mathrm{sech}(x)e^{\Gamma}, \tag{66}$$

and the interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \cosh(x)\sinh(x) + \left(1 - a_2C_0^2 - C_0^2V_2(x)\right)\mathrm{sech}^2(x) - a_1C_0\,\mathrm{sech}(x), \tag{67}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{68}$$

where

$$g(x) = \omega_2(\cosh^2(x)\,\mathrm{sech}^2(x)) - \frac{2\omega_1}{C_0^2}\left(\cosh^2(x)\ln(\cosh(x))\right). \tag{69}$$

For dark solitons we consider $C(x) = C_0\tanh(x)$ and we get

$$q(x,z) = C_0\tanh(x)e^{\Gamma}, \tag{70}$$

and the corresponding linear and nonlinear potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x)\right)\tanh^2(x) - a_1C_0\tanh(x), \tag{71}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{72}$$

where

$$g(x) = -2\omega_2\left(\sinh^2(x) - 2\coth^2(x)\ln(\cosh(x)) + 1\right) - \frac{\omega_1}{C_0^2}\left(\cosh^2(x) - 2\coth^2(x)\ln(\cosh(x))\right). \tag{73}$$

For singular solitons we consider $C(x) = C_0\coth(x)$ and we have

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{74}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) \\ -\Big(1 + a_2 C_0^2 + C_0^2 V_2(x)\Big)\coth^2(x) - a_1 C_0 \coth(x), \tag{75}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{76}$$

where

$$g(x) = -\omega_2\Big(\cosh^2(x) + 4\tanh^2(x)\ln(\sinh(x)) - 1\Big) - \\ \frac{\omega_1}{C_0^2}\Big(\sinh^2(x) + 2\ln(\sinh(x))\Big). \tag{77}$$

2.2.2. The imaginary parts of complex linear and nonlinear potentials having the form of product of sine and secant hyperbolic functions.

Here we consider the following $W_1(x)$ and $W_2(x)$.

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \tag{78}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{79}$$

In order to get bright solitons we consider $C(x) = C_0 \operatorname{sech}(x)$ and we get

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \tag{80}$$

and the interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \operatorname{sech}(x)\sinh(x) + \\ \Big(1 - a_2 C_0^2 - C_0^2 V_2(x)\Big)\operatorname{sech}^2(x) - a_1 C_0 \operatorname{sech}(x), \tag{81}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{82}$$

where

$$g(x) = -2\omega_2\left(\frac{\tanh^2(x)}{4} + \frac{\sinh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\sinh^2(x). \tag{83}$$

For dark solitons we consider $C(x) = C_0 \tanh(x)$ and we have

$$q(x,z) = C_0 \tanh(x) e^{\Gamma}, \tag{84}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x)\right)\tanh^2(x) - a_1C_0\tanh(x), \tag{85}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{86}$$

where

$$g(x) = -2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} + \frac{\tanh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2})\right). \tag{87}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no dark soliton in this physical model.

For singular solitons, we consider $C(x) = C_0\coth(x)$ and we get

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{88}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x)\right)\coth^2(x) - a_1C_0\coth(x), \tag{89}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{90}$$

where

$$g(x) = -2\omega_2\left(\ln(\sinh(x))\tanh^2(x) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{91}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no singular soliton in this physical model.

2.2.3. The imaginary part of linear potential is the product of hyperbolic cosine and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic secant and hyperbolic sine function.

In this Subsection, we take the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1\cosh(x)\sinh(x), \tag{92}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{93}$$

In order to get bright solitons we consider $C(x) = C_0 \operatorname{sech}(x)$ and we get

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \tag{94}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2} g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \cosh(x)\sinh(x) + \left(1 - a_2 C_0^2 - C_0^2 V_2(x)\right)\operatorname{sech}^2(x) - a_1 C_0 \operatorname{sech}(x), \tag{95}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{96}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x) + \frac{\sinh^2(x)}{4}\right) + \frac{-2\omega_1}{C_0^2}\cosh^2(x)\ln(\cosh(x)). \tag{97}$$

For dark solitons, we consider $C(x) = C_0 \tanh(x)$ and we get

$$q(x,z) = C_0 \tanh(x) e^{\Gamma}, \tag{98}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2} g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x)\right)\tanh^2(x) - a_1 C_0 \tanh(x), \tag{99}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{100}$$

where

$$g(x) = 2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} - \frac{\tanh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\left(\frac{\cosh^2(x)}{2} - \coth^2(x)\ln(\cosh(x))\right). \tag{101}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no dark soliton in this physical model.

For singular solitons, we consider $C(x) = C_0 \coth(x)$ and we have

$$q(x,z) = C_0 \coth(x)\Gamma, \tag{102}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2} g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x)\right)\coth^2(x) - a_1 C_0 \coth(x), \tag{103}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{104}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x)\ln(\sinh(x)) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\frac{\sinh^2(x)}{2} + \tanh^2(x)\ln(\sinh(x))\right). \tag{105}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no singular soliton in this physical model.

2.2.4. The imaginary part of linear potential is the product of hyperbolic secant and hyperbolic sine functions and the imaginary part of nonlinear potential is the product of hyperbolic cosine and hyperbolic sine functions.

In this Subsection, we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \tag{106}$$

$$W_2(x) = \omega_2 \cosh(x)\sinh(x), \tag{107}$$

For bright solitons, we consider $A(x) = A_0 \operatorname{sech}(x)$, then

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \tag{108}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \operatorname{sech}(x)\sinh(x) + \left(1 - a_2C_0^2 - C_0^2V_2(x)\right)\operatorname{sech}^2(x) - a_1C_0\operatorname{sech}(x), \tag{109}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{110}$$

where

$$g(x) = -\omega_2\sinh^2(x) - \frac{\omega_1}{C_0^2}\left(\sinh^2(x)\right). \tag{111}$$

For dark solitons, we consider $C(x) = C_0\tanh(x)$, then

$$q(x,z) = C_0\tanh(x)e^{\Gamma}, \tag{112}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1\operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x)\right)\tanh^2(x) - a_1C_0\tanh(x), \tag{113}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{114}$$

where

$$g(x) = -2\omega_2\left(1 - \ln(\cosh(x))\coth^2(x) + \sinh^2(x)\right) - \frac{2\omega_1}{C_0^2}\left(\cosh(x)\coth^2(x) - \frac{1}{2}\right). \tag{115}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no dark soliton in this physical model.

For singular solitons, we consider $C(x) = C_0\coth(x)$, then

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{116}$$

and the interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x)\right)\coth^2(x) - a_1C_0\coth(x), \tag{117}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{118}$$

where

$$g(x) = -2\omega_2\left(\cosh^2(x) + 2\ln(\sinh(x))\tanh^2(x) - 1\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{119}$$

According to the condition of PT-symmetry, $V_1(x)$ must be an even function. But here $V_1(x)$ is an odd function, so there is no singular soliton in this physical model.

2.3. ANTI-CUBIC LAW

In the case of anti-cubic law nonlinearity [35], we assume that $F(C) = a_1C^{-2} + a_2C + a_3C^2$, where a_1, a_2 and a_3 are constants. Thus Eq. (4) becomes

$$\frac{1}{2}\frac{d^2C}{dx^2} - \left(\frac{1}{2}g^2 + \lambda - V_1(x)\right)C + V_2(x)C^3 + a_1C^{-3} + a_2C^3 + a_3C^5 = 0. \tag{120}$$

2.3.1. The imaginary parts of linear and nonlinear potentials are the product of hyperbolic cosine and hyperbolic sine function.

Next, we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \cosh(x)\sinh(x), \tag{121}$$

$$W_2(x) = \omega_2 \cosh(x)\sinh(x), \tag{122}$$

For bright solitons, we consider $C(x) = C_0 \operatorname{sech}(x)$ and we get

$$q(x,z) = C_0 \operatorname{sech}(x)e^{\Gamma}, \tag{123}$$

and the associated potentials are given by

$$U_l(x) = -\left(1 + a_2C_0^2 + C_0^2V_2(x) + a_3C_0^4\operatorname{sech}^2(x)\right)\operatorname{sech}^2(x) + \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} - a_1C_0^{-4}\cosh^2(x) + i\omega_1\cosh(x)\sinh(x), \tag{124}$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \tag{125}$$

where

$$g(x) = \omega_2(\cosh^2(x)\operatorname{sech}^2(x)) - \frac{2\omega_1}{C_0^2}\left(\cosh^2(x)\ln(\cosh(x))\right). \tag{126}$$

For dark solitons, we consider $C(x) = C_0 \tanh(x)$ and we obtain

$$q(x,z) = C_0 \tanh(x)e^{\Gamma}, \tag{127}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1\cosh(x)\sinh(x) - a_1C_0^{-4}coth^2(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x) + a_3C_0^4\tanh^2(x)\right)\tanh^2(x), \tag{128}$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \tag{129}$$

where

$$g(x) = -2\omega_2\left(\sinh^2(x) - 2\coth^2(x)\ln(\cosh(x)) + 1\right) - \frac{\omega_1}{C_0^2}\left(\cosh^2(x) - 2\coth^2(x)\ln(\cosh(x))\right). \tag{130}$$

For singular solitons, we consider $C(x) = C_0 \coth(x)$, then

$$q(x,z) = C_0 \coth(x)e^{\Gamma}, \tag{131}$$

and associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 + i\omega_1 \cosh(x)\sinh(x) - a_1 C_0^{-4}\tanh^4(x) - \\ \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \coth^2(x)\right)\coth^2(x), \tag{132}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \cosh(x)\sinh(x), \tag{133}$$

where

$$g(x) = -\omega_2\left(\cosh^2(x) + 4\tanh^2(x)\ln(\sinh(x)) - 1\right) - \\ \frac{\omega_1}{C_0^2}\left(\sinh^2(x) + 2\ln(\sinh(x))\right). \tag{134}$$

2.3.2. The imaginary part of linear potential is the product of hyperbolic secant and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic secant and hyperbolic sine function.

Here we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \tag{135}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{136}$$

To get bright solitons, we consider $C(x) = C_0 \operatorname{sech}(x)$:

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \tag{137}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} + i\omega_1 \operatorname{sech}(x)\sinh(x) - a_1 C_0^{-4}\cosh^2(x) - \\ \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \operatorname{sech}^2(x)\right)\operatorname{sech}^2(x), \tag{138}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{139}$$

where

$$g(x) = -2\omega_2\left(\frac{\tanh^2(x)}{4} + \frac{\sinh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\sinh^2(x). \tag{140}$$

For dark solitons, we consider $C(x) = C_0 \tanh(x)$:

$$q(x,z) = C_0 \tanh(x) e^{\Gamma}, \tag{141}$$

and interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 - a_1 C_0^{-4}\coth^2(x) + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \tanh^2(x)\right)\tanh^2(x), \tag{142}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{143}$$

where

$$g(x) = -2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} + \frac{\tanh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2})\right). \tag{144}$$

For singular solitons, we consider $C(x) = C_0\coth(x)$:

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{145}$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 - a_1 C_0^{-4}\tanh^4(x) + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \coth^2(x)\right)\coth^2(x), \tag{146}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{147}$$

where

$$g(x) = -2\omega_2\left(\ln(\sinh(x))\tanh^2(x) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{148}$$

2.3.3. The imaginary part of linear potential is the product of hyperbolic cosine and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic secant and hyperbolic sine function.

Here we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1\cosh(x)\sinh(x), \tag{149}$$

$$W_2(x) = \omega_2 \operatorname{sech}(x)\sinh(x), \tag{150}$$

To get bright solitons, we consider $C(x) = C_0\operatorname{sech}(x)$:

$$q(x,z) = C_0\operatorname{sech}(x)e^{\Gamma}, \tag{151}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} - a_1 C_0^{-4}\cosh^2(x) + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C^4 \operatorname{sech}^2(x)\right)\operatorname{sech}^2(x), \tag{152}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{153}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x) + \frac{\sinh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\cosh^2(x)\ln(\cosh(x)). \tag{154}$$

For dark solitons, we consider $C(x) = C_0\tanh(x)$:

$$q(x,z) = C_0\tanh(x)e^{\Gamma}, \tag{155}$$

and the associated linear and nonlinear potentials potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 - a_1 C_0^{-4}\coth^2(x) + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \tanh^2(x)\right)\tanh^2(x), \tag{156}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{157}$$

where

$$g(x) = 2\omega_2\left(\coth^2(x)\ln(\cosh(x)) - \frac{1}{2} - \frac{\tanh^2(x)}{4}\right) - \frac{2\omega_1}{C_0^2}\left(\frac{\cosh^2(x)}{2} - \coth^2(x)\ln(\cosh(x))\right). \tag{158}$$

For singular solitons, we consider $C(x) = C_0\coth(x)$:

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{159}$$

and the corresponding potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 - a_1 C_0^{-4}\tanh^4(x) + i\omega_1 \cosh(x)\sinh(x) - \left(1 + a_2 C_0^2 + C_0^2 V_2(x) + a_3 C_0^4 \coth^2(x)\right)\coth^2(x), \tag{160}$$

$$U_{nl}(x) = V_2(x) + i\omega_2 \operatorname{sech}(x)\sinh(x), \tag{161}$$

where

$$g(x) = -2\omega_2\left(\tanh^2(x)\ln(\sinh(x)) - \frac{1}{2}\right) - \frac{2\omega_1}{C_0^2}\left(\frac{\sinh^2(x)}{2} + \tanh^2(x)\ln(\sinh(x))\right). \quad (162)$$

2.3.4. The imaginary part of linear potential is the product of hyperbolic secant and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic cosine and hyperbolic sine function.

Now we consider the following $W_1(x)$ and $W_2(x)$:

$$W_1(x) = \omega_1 \operatorname{sech}(x)\sinh(x), \quad (163)$$

$$W_2(x) = \omega_2 \cosh(x)\sinh(x), \quad (164)$$

To get bright solitons, we consider $C(x) = C_0 \operatorname{sech}(x)$:

$$q(x,z) = C_0 \operatorname{sech}(x) e^{\Gamma}, \quad (165)$$

and the interrelated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda - \frac{1}{2} - a_1 C_0^{-4}\cosh^2(x) + i\omega_1 \operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x) + a_3C^4\operatorname{sech}^2(x)\right)\operatorname{sech}^2(x), \quad (166)$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \quad (167)$$

where

$$g(x) = -\omega_2\sinh^2(x) - \frac{\omega_1}{C_0^2}\left(\sinh^2(x)\right). \quad (168)$$

For dark solitons, we consider $C(x) = C_0\tanh(x)$:

$$q(x,z) = C_0\tanh(x)e^{\Gamma}, \quad (169)$$

and the associated potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1a_1C_0^{-4}\coth^2(x) + i\omega_1\operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x) + a_3C_0^4\tanh^2(x)\right)\tanh^2(x) -, \quad (170)$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \quad (171)$$

where

$$g(x) = -2\omega_2\left(1 - \ln(\cosh(x))\coth^2(x) + \sinh^2(x)\right) - \frac{2\omega_1}{C_0^2}\left(\cosh(x)\coth^2(x) - \frac{1}{2}\right). \tag{172}$$

For singular solitons, we consider $C(x) = C_0\coth(x)$:

$$q(x,z) = C_0\coth(x)e^{\Gamma}, \tag{173}$$

and the interconnected potentials are

$$U_l(x) = \frac{1}{2}g^2(x) + \lambda + 1 - a_1C_0^{-4}\tanh^4(x) + i\omega_1\operatorname{sech}(x)\sinh(x) - \left(1 + a_2C_0^2 + C_0^2V_2(x) + a_3C_0^4\coth^2(x)\right)\coth^2(x), \tag{174}$$

$$U_{nl}(x) = V_2(x) + i\omega_2\cosh(x)\sinh(x), \tag{175}$$

where

$$g(x) = -2\omega_2\left(\cosh^2(x) + 2\ln(\sinh(x))\tanh^2(x) - 1\right) - \frac{2\omega_1}{C_0^2}\left(\tanh^2(x)\ln(\sinh(x))\right). \tag{176}$$

3. CONCLUSIONS

In this work, we have obtained families of exact solutions for optical solitons in PT-symmetric mixed linear-nonlinear optical lattices with both Kerr and non-Kerr law nonlinearities. We have considered four different types of complex-valued external potentials. First, we have considered that the imaginary parts of complex linear and nonlinear potentials have the form of the product of sine and cosine hyperbolic functions. Second, we have investigated the case when the imaginary parts of complex linear and nonlinear potentials have the form of the product of hyperbolic secant and hyperbolic sine functions. Third, we have considered the case when the imaginary part of linear potential is the product of hyperbolic cosine and hyperbolic sine functions and the imaginary part of nonlinear potential is the product of hyperbolic secant and hyperbolic sine functions. At the end, we have studied the case when the imaginary part of linear potential is the product of hyperbolic secant and hyperbolic sine function and the imaginary part of nonlinear potential is the product of hyperbolic cosine and hyperbolic sine function. We have used three different forms of optical nonlinearities: i) Kerr law nonlinearity gives bright, dark, and singular solitons, ii) quadratic-cubic law nonlinearity provides bright solitons, and iii) anti-cubic law nonlinearity gives bright, dark, and singular solitons.

Acknowledgements. The work of Qin Zhou was supported by the National Natural Science Foundation of China (Grant No. 11705130). Qin Zhou was also sponsored by the Chutian Scholar Program of Hubei Government in China.

REFERENCES

1. C. M. Bender and S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having PT symmetry*, Phys. Rev. Lett. **80**, 5243–5246 (1998).
2. V. V. Konotop, J. Yang, and D. A. Zezyulin, *Nonlinear waves in PT-symmetric systems*, Rev. Mod. Phys. **88**, 035002 (2016).
3. Y. V. Kartashov, B. A. Malomed, and L. Torner, *Solitons in nonlinear lattices*, Rev. Mod. Phys. **83**, 247–306 (2011).
4. V. A. Brazhnyi and V. V. Konotop, *Theory of nonlinear matter waves in optical lattices*, Mod. Phys. Lett. B **18**, 627-651 (2004).
5. D. A. Zezyulin and V. V. Konotop, *Nonlinear Modes in Finite-Dimensional PT-Symmetric Systems*, Phys. Rev. Lett. **108**, 213906 (2012).
6. F. K. Abdullaev, Y. V. Kartashov, V. V. Konotop, and D. A. Zezyulin, *Solitons in PT-symmetric nonlinear lattices*, Phys. Rev. A **83**, 041805 (2011).
7. B. A. Malomed and D. Mihalache, *Nonlinear waves in optical and matter-wave media: A topical survey of recent theoretical and experimental results*, Rom. J. Phys. **64**, 106 (2019).
8. P. Li, J. Li, B. Han, H. Ma, and D. Mihalache, *PT-symmetric optical modes and spontaneous symmetry breaking in the space-fractional Schrödinger equation*, Rom. Rep. Phys. **71**, 106 (2019).
9. H. Wang, X. P. Ren, D. Mihalache, Y. H. Weng, J. Huang, and Y. J. He, *Defect modes supported by parity-time-symmetric triangular optical lattices with self-defocusing Kerr nonlinearity*, Rom. Rep. Phys. **71**, 411 (2019).
10. Y. He, X. Zhu, D. Mihalache, J. Liu, and Z. Chen, *Solitons in PT-symmetric optical lattices with spatially periodic modulation of nonlinearity*, Opt. Commun. **285**, 3320–3324 (2012).
11. Y. He, X. Zhu, D. Mihalache, J. Liu, and Z. Chen, *Lattice solitons in PT-symmetric mixed linear-nonlinear optical lattices*, Phys. Rev. A **85**, 013831 (2012).
12. R. J. Lombard, R. Mezhoud, and R. Yekken, *Complex potentials with real eigenvalues and the inverse problem*, Rom. J. Phys. **63**, 101 (2018).
13. Y. He and D. Mihalache, *Spatial solitons in parity-time-symmetric mixed linear-nonlinear optical lattices: Recent theoretical results*, Rom. Rep. Phys. **64**, 1243–1258 (2012).
14. P. Li, D. Mihalache, and L. Li, *Asymmetric solitons in parity-time-symmetric double-hump Scarff-II potentials*, Rom. J. Phys. **61**, 1028–1039 (2016).
15. P. Li, D. Mihalache, and B. A. Malomed, *Optical solitons in media with focusing and defocusing saturable nonlinearity and a parity-time-symmetric external potential*, Phil. Trans. R. Soc. A **376**, 20170378 (2018).
16. P. Li, B. Liu, L. Li, and D. Mihalache, *Nonlinear parity-time symmetric breaking optical waveguides with complex Gaussian-type potential*, Rom. J. Phys. **61**, 577–594 (2016).
17. P. Li, L. Li, and D. Mihalache, *Optical solitons in PT-symmetric potentials with competing cubic-quintic nonlinearity: Existence, stability, and dynamics*, Rom. Rep. Phys. **70**, 408 (2018).
18. N. Bender, S. Factor, J. D. Bodyfelt, H. Ramezani, D. N. Christodoulides, F. M. Ellis, and T.

Kottos, *Observation of asymmetric transport in structures with active nonlinearities*, Phys. Rev. Lett. **110**, 234101 (2013).

19. J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, *Experimental study of active LRC circuits with PT symmetries*, Phys. Rev. **84**, 040101 (2011).
20. A. Mostafazadeh, *Pseudo-Hermiticity versus PT symmetry: the necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian*, J. Math. Phys. **43**, 205–214 (2002).
21. S. Bittner, B. Dietz, U. Gunther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schafer, *PT symmetry and spontaneous symmetry breaking in a microwave billiard*, Phys. Rev. Lett. **108**, 024101 (2012).
22. Y. D. Chong, L. Ge, and A. D. Stone, *Symmetry breaking and laser-absorber modes in optical scattering systems*, Phys. Rev. Lett. **106**, 093902 (2011).
23. Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, *Optical solitons in PT periodic potentials*, Phys. Rev. Lett. **100**, 030402 (2008).
24. Y. Lumer, Y. Plotnik, M. C. Rechtsman, and M. Segev, *Nonlinearly induced PT transition in photonic systems*, Phys. Rev. Lett. **111**, 263901 (2013).
25. H. Wang, S. Shi, X. Ren, X. Zhu, B. A. Malomed, D. Mihalache, and Y. He, *Two-dimensional solitons in triangular photonic lattices with parity-time symmetry*, Opt. Commun. **335**, 146–152 (2015).
26. X. Zhu and H. Li, *Multihump solitons in two-dimensional parity-time-symmetric optical lattices with focusing saturable nonlinearity*, Eur. Phys. J. D **70**, 14 (2016).
27. Y. Y. Wang, L. Chen, C. Q. Dai, J. Zheng, and Y. Fan, *Exact vector multipole and vortex solitons in the media with spatially modulated cubic-quintic nonlinearity*, Nonlinear Dyn. **90**, 1269–1275 (2017).
28. F. K. Abdullaev, V. V. Konotop, M. Salerno, and A. V. Yulin, *Dissipative periodic waves, solitons, and breathers of the nonlinear Schrödinger equation with complex potentials*, Phys. Rev. E **82**, 056606 (2010).
29. D. Nath, B. Roy, and R. Roychoudhury, *PT symmetric nonlinear optical lattice: analytical solutions*, Chaos Soliton. Fract. **81**, 91–97 (2015).
30. Y. X. Chang, F. Q. Xu, and Y. L. Hu, *Bright spatial solitons in quintic-septimal non-linear media with two families of PT-symmetric potentials*, Eur. Phys. J. Plus **132**, 533 (2017).
31. C. Q. Dai, Y. Fan, and Y. Y. Wang, *Three-dimensional optical solitons formed by the balance between different-order nonlinearities and high-order dispersion/diffraction in parity-time symmetric potentials*, Nonlinear Dyn. **98**, 489–499 (2019).
32. A. Zubair, N. Raza, M. Mirzazadeh, W. Liu, and Q. Zhou, *Analytic study on optical solitons in parity-time-symmetric mixed linear and nonlinear modulation lattices with non-Kerr nonlinearities*, Optik **173**, 249–262 (2018).
33. Q. Zhou, *Analytic study on optical soliton in a Kerr-law medium with an imprinted parity-time-symmetric mixed linear-nonlinear lattice*, Proceedings of the Romanian Academy, Series A **18**, 223–230 (2017).
34. S. Kumar, Q. Zhou, and W. Liu, *Invariant traveling wave solutions of parity-time-symmetric mixed linear-nonlinear optical lattices with three types of nonlinearity*, Laser Phys. **29**, 045401 (2019).
35. A. Biswas, M. Ekici, A. Sonmezoglu, S. Arshed, and M. Belic, *Optical soliton perturbation with full nonlinearity by extended trial function method*, Optical and Quantum Electron. **50**, 449 (2018).