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ON THE PHASE SHIFTS OF THE (UN)MODULATED DUST KINETIC ALFVÉN SOLITONS COLLISIONS IN A THERMAL DISTRIBUTED NORMAL PLASMA

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Abstract. The collisions between two unmodulated (waves that propagate at a phase velocity) and modulated (waves that propagate at a group velocity) dust kinetic Alfvén solitons (DKASs) have been analyzed in a low-β dusty plasma consisting of inertial negatively dust charged grain in addition to Boltzmann-Maxwellian distributed electrons and ions. In this work we study two important types of solitons collision. The first one is the unmodulated solitons collision. For studying the collisions between the unmodulated solitons, the extended Poincaré-Lighthill-Kuo (PLK) technique has been adopted to derive the two Korteweg-de Vries (KdV) equations for the right and left moving solitons and their phase shifts. The analysis showed that for small amplitudes, *i.e.*, weakly nonlinearity and quasi-elastic collisions, the colliding solitons amplitude at the colliding point nearly equals to the sum of individual amplitudes. Moreover, the impact of the related physical parameters on the phase shifts is examined. In the second part of this paper, the KdV equation is converted to the nonlinear Schrödinger equation (NLSE) for investigating the modulated dark solitons. Then for studying the dark soliton collisions, the PLK technique is introduced. Furthermore, the impact of relevant plasma parameters on the dark soliton phase shift is reported. This work may apply to understand the creation of nonlinear coherent structures and their collisions in astrophysical and space plasma domain where dust grains exist.

Key words: Dust Alfvén solitons, head-on collision, PLK method, dark Solitons.

1. INTRODUCTION

It is known that the propagation direction of kinetic Alfvén waves (KAWs) is almost perpendicular to the ambient magnetic field. The dispersive KAWs play an essential role in explaining electromagnetic fluctuations, particle accelerations, and nonlinear waves existing in astrophysical, space, and laboratory plasmas [1]. The dispersion effects in KAWs come *via* two different effects: (i) the first when the ion gyroradius is equal in the perpendicular wavelength (*i.e.*, $k_\perp \simeq \rho_i$) in hot electron plasma and in this case is known as KAW and (ii) the second when the electron inertial length is comparable to the perpendicular wavelength (*i.e.*, $k_\perp \simeq \lambda_i$) in the cold electron plasma which is called the shear Alfvén wave [2] . Hasegawa and Mima had examined analytically the KAWs in a low β plasma such as $(m_e/m_i << \beta << 1)$ in collisionless electron-ion plasmas, where $\beta = (4\pi n_{i0} T_e)/B_0^2 \equiv c_s^2/v_A^2$. Here, the parameters $m_e\,(m_i)$ represents the electron (ion) mass, T_e, n_{i0}, and B_0 are the electron temperature, unperturbed ion density, and the external magnetic field, while the acoustic speed and Alfvén speed are, respectively, defined as $c_s = (T_e/m_i)^{1/2}$ and $v_A = \left(B_0^2/(4\pi n_{i0} m_i)\right)^{1/2}$.

The data from the Freja and FAST satellites shows that the auroral low-frequency turbulence is prevailed by intense electromagnetic bursts, density cavities in the earth's ionosphere's auroral region that may be like KAW solitary structures [3–5]. They are known as the unmodulated solitons (waves propagate with phase velocity) that are created as a result of the balance between dispersion and nonlinearity effects. They maintain their structure after collisions with another solitons of the same kind and are only deviated from their trajectories, *i.e.*, phase shifts occur [6]. These waves can be described using a huge amount of partial differential equations (PDEs) that can describe these types of waves such as the Korteweg–de Vries (KdV) equation, the modified KdV equation, Gardner equation, higher-order KdV equations, the Kadomtsev–Petviashvili (KP) equation and its higher-order, the Zakharov-Kuznetsov equation and its higher-order, etc. [7–9]. The above mentioned equations can be obtained by reducing the basic equations of different plasma models using the reductive perturbation method (RPM) [10–15]. The extended Poincaré-Lighthill-Kuo (PLK) technique is adopted, for studying the head-on collision between solitons [10–16]. One of the basic functions of this method is to reduce the basic set of fluid equations of the plasma model to the two evolution equations with the two phase shifts due to the waves collision [17–23]. It should be noted here that the extended PLK method is only valid for small amplitudes of the colliding waves [24]. There are two types of unmodulated solitons collisions, *i.e.*, (i) interaction of solitons propagating in the same direction that is called overtaking collision and in this case the velocities of solitons must be different in order to achieve collision [25] and (ii) interaction of oppositely travelling solitons known as head-on collision (HOC) [26]. In this regard, a tremendous number of papers have been published about the study of the unmodulated solitons collisions in different plasma models. To name a few examples, the dust-acoustic solitons (DASs) collisions in a strongly coupled complex plasma has been investigated experimentally [27]. The authors found great agreement between the theoretical results and laboratory observations that validated the theoretical results. The HOC between dust-acoustic multi-solitons (DAMSs) in a complex plasma consisting of inertial dust particles and inertialess Maxwellian electron, and two various types of non-Maxwellian positive ions (nonextensivelly for cold ion and nonthermality for hot ion) is examined using the extended PLK [28]. The impact of polarization force on the profiles of the colliding DAMSs in a complex plasma having non-Maxwellian ion has been studied [29]. Moreover, in a complex plasma composed of dust fluid and superthermal species (electron, ion, and positron), the HOC among two DASs and the rogue waves (RWs) propagation in the framework of KdV equations has been investigated [30].

Many types of localized wave structures that form and are stable in a series of physical settings have been studied during recent years; see, for example, two extensive review papers [31, 32]. Over the past years, the interesting collision scenarios and interaction between such localized wavefronts in both conservative and dissipative systems have been studied in many physical contexts; see, for example, Refs. [33–41].

In a magnetized plasma, the interaction between two oppositely magnetoacoustic solitons is reported in a collisionless magneto plasma consisting of inertial cold ion and warm electron [42]. Moreover, the interaction between two kinetic Alfvén (KA) unmodulated solitons in electron-ion plasma with kappa distributed electrons has been reported [43]. There is another type of solitons that can propagate in the nonlinear and dispersive medium such as ocean, optical fibres, and plasma physics, which is called modulated envelope soliton. This type of waves stems from balancing the nonlinearity (self-steepening) and group dispersion [44, 45]. Here, the dynamics of the modulated envelope solitons are governed by the well known NLSE and its higher-order family. There are many mathematical methods that are used to obtain these PDEs (the NLSE and its family) from the fluid model and these methods come from manifold from a reductive perturbation technique (RPT) such as multiple scale [46], the derivative expansion method [47–49], and the Krylov-Bogoliubov-Mitropolsky (KBM) method [50, 51]. It is known that the scenario of the modulated envelope waves propagation in a dispersive and nonlinear medium (such as plasmas) is subjected to the modulational instability (MI). So, depending on the study of MI, we can determine the types (bright-, dark-, gray-solitons, and breathers waves) and regions of modulated envelope wave propagation in different plasma media. It should be noted here that if the NLSE is obtained from the basic fluid equations of any plasma model, we finally get an equation with an arbitrary wavenumber. Under this case, the modulated structures may be stable or unstable depending on relevant physical plasma parameters. On the contrary, if we derive the NLSE from the KdV equation, we finally obtain an equation with a small wavenumber and in this case only stable modulated wave (dark solitons) can propagate in the plasma model [52–59]. However, the matter is different with regard to the mKdV equation and the EKdV equation, where the regions of stable and unstable wave package propagation depend on system parameters [52–59]. In the last decade the collision of modulated solitons (dark solitons) attract much attention for their interesting properties [60]. Many authors have studied head-on (face-to-face) collision of dark solitons in many nonlinear media such as optical fibers [61, 62], Bose-Einstein condensates (BECs) [63], and in plasma scenarios [64].

The occurrences of dust charge grains in the earth's mesosphere, interstellar clouds, cometary tails, ionosphere, in the rings of Jupiter and Saturn makes it crucial to examine [65]. In the laboratory, the existence of dusty (complex) plasma is obvious through some mechanisms and detail is found in Refs. [66, 67]. The existence of charged dust grains in any plasma model modifies both the KAWs dispersion relation and the wave modes that can exist and propagate in the plasma model *e.g.*, dust kinetic Alfvén waves (DKAWs) [68]. For instance, Mahmood *et al.* [69] analyzed the dust kinetic shear Alfvén wave (DKSAWs) in a complex plasma having thermal electron and ion. They found that both density positive polarity (hump) and negative polarity (dip) solitons travelling at either sub-Alfvénic or super-Alfvénic velocities coexist. Yinhua *et al.* [70] investigated the (non)linear characteristics of dust kinetic Alfvén waves (DKAWs) in a cold dusty plasma with Maxwellian electron and ion. They found that the amplitudes of such dust kinetic Alfvén solitons (DKASs) are significantly modified by the dust density variation. Mirza *et al.* [71] studied the nonlinear characteristic of DKASs in two-components plasmas and they observed that both compressive (positive) and rarefactive (negative) DKASs can exist at sub-Alfvénic speed, while at super-Alfvénic speed only positive DKASs can propagate. Mahmood and Saleem [72] examined the characteristic of both dust-acoustic waves (DAWs) and DKAWs in both a complex plasma consisting only from inertial dust and inertial positive ion and in another complex plasma model composed of inertial dust and inertialess ion and electron. The authors discussed the effect of plasma β on the soliton profile in the two cases and observed that the plasma β has an effective effect on both DAWs and DKAWs. Bains *et al.* [73] have reported the kinetic Alfvén solitary waves (KASWs) and kinetic Alfvén rogue waves (KARWs) in a superthermal normal low beta plasma. Saini *et al.* [74] examined the characteristic of DKASs in low β plasma. They numerically analyzed the influence of the dust concentration, obliqueness, superthermal index κ, and plasma β on the amplitude and width of the solitons. Adnan *et al.* [75] have reported the existence of the sub-Alfvénic compressive solitons in e-p-i plasma. They studied the effects of the plasma

β, superthermality of electrons and positrons on the shape and size of KA compressive solitons. Ahmed and Sah [76] considered nonextensive electrons, nonextensive positrons, and ions. They studied the impact of different parameters on the properties of solitary KAWs.

Under the assumption of low frequency on the timescale of the dust grains, the dust kinetic Alfvén waves (DKAWs) can be generated if both dust grains *via* their polarization drift interact with the transverse perturbation magnetic field. On the other hand, the light particles such as electrons and ions are treated as inertialess particles obeying the Maxwellian distribution. Moreover, the pressures of these light particles that can act on the fluid dust grains through the self-consistent electrostatic field cause gyromotion [77]. In the dusty plasma having inertialess electrons and ions, the dust-acoustic speed and dust-Alfvén speed are, respectively, $v_{sd} = (T_{eff}/m_d)^{1/2}$ and $v_{Ad} = \left(B_0^2/(4\pi n_{d0} m_d)\right)^{1/2}$, where $T_{eff} = \left(Z_d^2 n_{d0} T_i T_e\right)/(n_{i0} T_e + n_{e0} T_i)$ gives the effective temperature. In the present manuscript we assume that $v_{sd} << v_{Ad}$ and $\beta_d = \frac{4\pi n_{d0} T_{eff}}{B_0^2} << 1$, *i.e.*, $m_i/m_d << \beta_d << 1/Z_d$. Here, Z_d represents the amount of charge resides on the surface of the dust particle.

To the best of our knowledge, the face-to-face (head-on) collisions between two unmodulated and modulated dust kinetic Alfvén solitons (DKASs) have not been examined till now in a complex magnetoplasma. Thus, in this manuscript, we will investigate two issues. The first one is studying the unmodulated soliton collisions by reducing the fluid equations of the model using PLK method to two counterpart KdV equations with their phase shifts. The second issue is to investigate the modulated envelope dark soliton collisions. In the later case, the KdV equation will convert to the NLSE at small wavenumber using derivative expansion method. After that the PLK technique is introduced for investigating the envelope dark soliton collisions.

This paper is organized as follows. The physical model for DKAWs is presented and the linear dispersion relation is obtained in Sec. 2. In Sec. 3, the mechanism of the face-to-face collisions between the two counterpart propagating DKA solitons and their related phase shifts are analytically deduced using the extended PLK technique and collisions of unmodulated structures are investigated. The collisions between two modulated (envelope) dark solitons and their dynamics are analytically investigated in Sec. 4. Finally, in Sec. 5 the conclusions are presented.

2. MODEL EQUATIONS OF THE PHYSICAL PROBLEM

We consider that the three component plasma consists of negative dust charged grains, ions and electrons in a small but finite β_d collisionless magnetized homogeneous plasma. Under the low frequency condition such as $\partial/\partial t << \Omega_d$ ($\Omega_d = Z_d e B_0/m_d c$, dust cyclotron frequency) in xz plane, the dynamics of dust kinetic

Alfvén wave is described by the following continuity equation:

$$\partial_t n_d + \partial_x (n_d v_{dx}) = 0. \tag{1}$$

The electric field E can be expressed as the gradient of two scalar potentials ϕ and ψ *i.e.*, $E_x = -\partial\phi/\partial x, E_z = -\partial\psi/\partial z, E_y = 0$, along x-axis the dust motion is governed by the polarization drift, *i.e.*,

$$v_{dx} = \frac{Z_d e B_0}{m_d \Omega_d^2} \partial^2_{x,t} \phi. \tag{2}$$

In the strongly magnetized dust plasma, the dust parallel velocity v_{dz} along the external magnetic field B_0 has been ignored:

$$\partial_x B_y = \frac{4\pi J_z}{c}, \tag{3}$$

where the parallel current density is $J_z = e(n_i v_{iz} - n_e v_{ez})$. The combination between Faraday's law and Ampere's law reads as

$$\partial^4_{x^2,z^2}(\phi - \psi) = -\frac{4\pi Z_d e}{c} \partial^2_{t^2} n_d. \tag{4}$$

The Maxwell-Boltzmann density of electron and ion is written as

$$n_e = n_{e0} \exp\left(\frac{e\psi}{T_e}\right), \tag{5}$$

$$n_i = n_{i0} \exp\left(\frac{-e\psi}{T_i}\right). \tag{6}$$

The quasi-neutrality condition reads as

$$n_e + Z_d n_d - n_i \simeq 0, \tag{7}$$

the current density continuity is given by

$$\partial_z J_z = e\partial_t (n_e - n_i), \tag{8}$$

$$\partial_z J_z = -\partial_t (Z_d n_d e). \tag{9}$$

By considering that all the perturbed physical parameters are proportional to $\exp[\iota(k_x x - k_z z - \omega t)]$, henceforth by solving the Eqs. (1)-(9) yields the following linear dispersion relation

$$\omega^2 = k_z^2 v_{Ad}^2 (1 + \rho_d^2 k_x^2). \tag{10}$$

Here $\rho_d = c_{sd}/\omega_d$ is the dust gyroradius and k_x, k_z, and c_{ds} are the wavenumber vector along x, z direction, and dust acoustic speed respectively. In Eq. (10), dispersion arises due to the effect of dust gyroradius *i.e.*, the DKAWs propagate obliquely in the direction of an ambient magnetic field. Equation (10) is identical as Eq. (14) of Yinhua *et al.* [77]. The normalized set of dynamical equations (1)-(9) can be introduced in the following way

$$\partial_x n_d + \partial_x (n_d v_{dx}) = 0, \tag{11}$$

$$v_{dx} = \beta_d \partial^2_{x,t} \Phi. \tag{12}$$

The Ampere's and Faraday's laws can be combined in the following form

$$\partial^4_{x^2,z^2}(\Phi - \Psi) = -\frac{1}{\beta_d}\partial^2_{t^2} n_d. \tag{13}$$

The normalized electron and ion densities read as

$$n_e = \exp\left(\frac{\sigma\Psi}{\alpha}\right), \tag{14}$$

$$n_i = \exp\left(\frac{-\Psi}{\alpha}\right). \tag{15}$$

Finally, the normalized neutrality condition reads as

$$n_d = (1-\delta) n_i - \delta n_e. \tag{16}$$

In Eqs. (12)-(15) we adopted the following normalization schemes as, the time and space, respectively, are scaled as $t \to \Omega_d^{-1} t$, and $x(z) \to \frac{c}{\omega_{pd}} x(z)$, the dust fluid velocity is normalized as $v_{dx} \to v_{dx} v_{Ad}$, and the electron and ion densities are scaled as $n_j \to n_j / n_{j0}$ with $j = d, e$, and i, where n_{j0} gives the unperturbed number density, and the dust plasma frequency is defined as $\omega_{pd} = (4\pi Z_d^2 n_d e^2/m_d)^{1/2}$.

The electrostatic potentials are normalized as $(\Phi, \Psi) \to e(\phi, \psi) T_{eff}/e$, where $\delta = n_{e0}/(Z_d n_{d0})$, $\sigma = T_i/T_e$, $\alpha = T_i/T_{eff}$, $1+\delta = n_{i0}/(Z_d n_{d0})$, and $n_e = n_i^{-\sigma}$. The tilda signs on normalized variables have been removed for simplicity.

3. COLLISIONS OF UNMODULATED STRUCTURES

For studying the dust kinetic Alfvén soliton collisions of small but finite amplitude in magnetized homogeneous collisionless plasma, we used the extended PLK method [22, 23]. Accordingly, the stretching of space and time coordinate is given as

$$\begin{aligned}
\zeta &= \epsilon^{\frac{1}{2}}(l_x x + l_z z - \lambda t) + \epsilon \Theta_R(\zeta, \chi, \tau), \\
\chi &= \epsilon^{\frac{1}{2}}(l_x x + l_z z + \lambda t) + \epsilon \Theta_L(\zeta, \chi, \tau), \\
\tau &= \epsilon^{\frac{3}{2}} t.
\end{aligned} \tag{17}$$

The variables ζ (χ) symbolizes the trajectory of the right (left) moving soliton. In system (17) Θ_R and Θ_L are the unknown phase functions and λ is the phase velocity (scaled by dust Alfvén speed (v_{Ad})), to be determined later by removing the secularities from the higher order perturbations. The symbols l_x and l_z represent the direction cosine such that $l_x^2 + l_z^2 = 1$. The following operators for space and time are introduced

$$\left(\hat{X}_1, \hat{Z}_1\right) = (l_x, l_z)\left(\partial_\zeta + \partial_\chi\right),$$
$$\left(\hat{X}_2, \hat{Z}_2\right) = \left(\hat{X}_1, \hat{Z}_1\right)\Theta_R\partial_\zeta + \left(\hat{X}_1, \hat{Z}_1\right)\Theta_L\partial_\chi$$
$$\hat{T}_1 = \lambda\left(-\partial_\zeta + \partial_\chi\right), \ \& \ \hat{T}_2 = \hat{T}_1\Theta_R\partial_\zeta + \hat{T}_1\Theta_L\partial_\chi. \tag{18}$$

The two oppositely propagating DKA solitons (DKASs) are initially far away from one another, and after some time interval they are colliding and in the interaction region a new soliton is created and after collision, the running left soliton is moving towards the right and the right travelling soliton is moving toward the left with a slight change in the trajectories after the collision, which is called the phase shift. The dependent quantities are perturbed around their unperturbed values as

$$n_d = 1 + \epsilon n_{d1} + \epsilon^2 n_{d2} + ..., \tag{19}$$
$$v_{dx} = \epsilon v_{d1} + \epsilon^2 v_{d2} + ..., \tag{20}$$
$$\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + ..., \tag{21}$$
$$\Phi = \Phi_1 + \epsilon\Phi_2 + ... \tag{22}$$

The strength of the nonlinearity and dispersion is given by a dimensionless parameter ϵ *i.e.*, $(0 < \epsilon << 1)$. The quantities l_x and l_z give the cosines directional and they satisfy $l_x^2 + l_z^2 = 1$.

Now, by using Eqs. (19)-(22) with operators (18) in Eqs. (11)-(13), and collecting lowest-order coefficients of ϵ, *i.e.*, ($\sim \epsilon^{3/2}$ and ϵ) we have

$$\widehat{T}_1 n_{d1} + \widehat{X}_1 v_{dx1} = 0, \tag{23}$$
$$v_{dx1} = \beta\widehat{X}_1\widehat{T}_1\Phi_1, \tag{24}$$
$$\widehat{X}_1^2\widehat{Z}_1^2 = -\frac{1}{\beta_d}\widehat{T}_1^2 n_{d1}, \tag{25}$$
$$n_{d1} = -\gamma\Psi_1. \tag{26}$$

Here, $\gamma = (1 + \delta + \delta\sigma)/\alpha$. The solution of Eqs. (23)-(26), gives the following relation

$$(l_z^2 - \lambda^2)\left(\partial_\zeta^2\Psi - \partial_\chi^2\Psi\right) + 2(l_z^2 + \lambda^2)\partial_{\zeta,\chi}^2\Psi = 0. \tag{27}$$

For the dust kinetic Alfvénic mode, *i.e.*, $\lambda^2 = l_z^2$, the relation (27) reduces to

$$4l_z^2\partial_{\zeta,\chi}^2\Psi = 0. \tag{28}$$

Equation (28) can be classified as hyperbolic equation whose solution is the sum of two variables or functions:

$$\Psi_1 = \Psi_{1\zeta} + \Psi_{1\chi}, \tag{29}$$

where $\Psi_{1\zeta} \equiv \Psi_1(\zeta,\tau)$ and $\Psi_{1\chi} \equiv \Psi_1(\chi,\tau)$ and with the aid of Eq. (29), the solution of Eqs. (23)-(26) yields

$$n_{d1} = -\gamma(\Psi_{1\zeta} + \Psi_{1\chi}), \tag{30}$$

$$v_{dx} = -\gamma\frac{\lambda}{l_x}(\Psi_{1\zeta} - \Psi_{1\chi}), \tag{31}$$

$$\left(\partial^2_{\zeta^2} + \partial^2_{\chi^2}\right)^2 \Phi = \frac{\gamma}{\beta_d l_x^2}(\Psi_{1\zeta} + \Psi_{1\chi}). \tag{32}$$

Taking now the next higher-order terms of ϵ ($\sim \epsilon^{5/2}$ and ϵ^2), we get

$$\widehat{T}_1 n_{d2} + \widehat{T}_2 n_{d1} + \partial_\tau n_{d1} + \widehat{X}_1 v_{dx2} + \widehat{X}_2 v_{dx1} + \widehat{X}_1 n_{d1} v_{dx1} = 0, \tag{33}$$

$$v_{dx2} = \beta_d \widehat{X}_1 \widehat{T}_1 \Phi_2 + \beta_d \widehat{X}_1 \widehat{T}_2 \Phi_1 + \beta_d \widehat{X}_1 \partial_\tau \Phi_1 + \beta_d \widehat{X}_2 \widehat{T}_1 \Phi_1, \tag{34}$$

$$\widehat{X}_1^2 \widehat{Z}_1^2 (\Phi_2 - \Psi_1) = -\frac{1}{\beta_d}\left(\widehat{T}_1 n_{d2} + 2\widehat{T}_1\widehat{T}_2 n_{d1} + 2\widehat{T}_1 \partial_\tau n_{d1}\right), \tag{35}$$

$$n_{d2} = -\gamma\Psi_2 + \mu\Psi_1^2, \tag{36}$$

where $\mu = \left(1+\delta-\delta\sigma^2\right)/\left(2\alpha^2\right)$. Eliminating the second-order parameters in Eqs. (33)-(36), we have

$$\begin{aligned}
-4\gamma\lambda^2\widehat{T}_1\partial^2_{\zeta,\chi}\Psi_2 &= \widehat{Z}_1^2\widehat{T}_2 n_{d1} - \widehat{Z}_1^2\partial_\tau n_{d1} - \widehat{Z}_1^2\widehat{X}_1 n_{d1} v_{dx1} - \widehat{Z}_1^2\widehat{X}_2 v_{dx1} \\
&-\beta_d\widehat{Z}_1^2\widehat{X}_1^2\widehat{T}_1\Phi_1 - \beta_d\widehat{Z}_1^2\widehat{X}_1\partial_\tau\Phi_1 + \beta_d\widehat{Z}_1^2\widehat{X}_2\widehat{T}_1\Phi_1 - \widehat{T}_1\left(\widehat{Z}_1^2 - \widehat{T}_1^2\right)\mu\Psi_1^2 \\
&-\beta_d\widehat{T}_1\widehat{X}_1^2\widehat{Z}_1^2\Psi_1 + 2\widehat{T}_1\widehat{T}_2 n_{d1} + 2\widehat{T}_1\partial_\tau n_{d1}.
\end{aligned} \tag{37}$$

With the help of the first-order solutions, one can simplify Eqs. (37) as follows

$$\begin{aligned}
-\partial^2_{\zeta,\chi}\Psi_2 &= \frac{1}{2\lambda}\partial_\zeta\left(\partial_\tau\Psi_{1\zeta} + A\Psi_{1\zeta}\partial_\zeta\Psi_{1\zeta} + B\partial_\zeta^3\Psi_{1\zeta}\right) \\
&- \frac{1}{2\lambda}\partial_\chi\left(\partial_\tau\Psi_{1\chi} - A\Psi_{1\chi}\partial_\chi\Psi_{1\chi} - B\partial_\chi^3\Psi_{1\chi}\right) \\
&+ \frac{1}{4}\left(\partial_\chi\Theta_R + C\Psi_{1\chi}\right)\partial_\zeta^2\Psi_{1\zeta} - \frac{1}{4}\left(\partial_\zeta\Theta_L + C\Psi_{1\zeta}\right)\partial_\chi^2\Psi_{1\chi}.
\end{aligned} \tag{38}$$

The first two terms on the RHS of Eq. (38) will give the secular terms at this order and at the next higher-order, the third and fourth terms will generate secularities, hence these secularities should be eliminating. Thus, one finally obtains

$$\partial_\tau\Psi_{1\zeta} + A\Psi_{1\zeta}\partial_\zeta\Psi_{1\zeta} + B\partial_\zeta^3\Psi_{1\zeta} = 0, \tag{39}$$

$$\partial_\tau\Psi_{1\chi} - A\Psi_{1\chi}\partial_\chi\Psi_{1\chi} - B\partial_\chi^3\Psi_{1\chi} = 0. \tag{40}$$

and

$$\partial_\chi\Theta_L - C\Psi_{1\chi} = 0, \tag{41}$$

$$\partial_\zeta\Theta_R - C\Psi_{1\zeta} = 0. \tag{42}$$

Here, $A = l_z(1+\delta+\delta\alpha)$, $B = \left(-\beta_d l_x^2 lz/\gamma\right)$, and $C = 2\gamma$. Equations (39) and (40) represent the KdV equations of the right and left going solitons, respectively, whose solutions are given as

$$\Psi_{1\zeta} = \Psi_R \mathrm{sec\,h}^2 \sqrt{\left(\frac{u_R}{4B}\right)} (\zeta - u_R\tau), \tag{43}$$

$$\Psi_{1\chi} = \Psi_L \mathrm{sec\,h}^2 \sqrt{\left(\frac{v_L}{4B}\right)} (\chi - v_L\tau), \tag{44}$$

where $\Psi_R = (3u_R/A)$ and $\Psi_L = (3v_L/A)$ are the amplitude of the right (R) and left (L) going solitons, respectively, and $W_R = \sqrt{4B/u_R}$ and $W_L = \sqrt{4B/v_L}$ give the width of right (R) and left (L) going solitons, respectively. While u and v are, respectively, the initial velocities of R and L. Note that due to the balance and interaction between the nonlinearity and the dispersion, the unmodulated soliton structure can be created. It should be noted something important here: as $A \propto l_z$ and $B \propto l_x^2 l_z$; so at $\theta = 0°$, $A \neq 0$ but $B = 0$ and at $\theta = 90°$ both coefficients vanish. Thus, we consider the value of an angle θ as $0° < \theta < 90°$. The solution of Eqs. (41) and (42) yield as

$$\Theta_L = \frac{AC}{6\sqrt{Bv_L}} \left[\tanh \sqrt{\left(\frac{v_L}{4B}\right)} (\chi + v_L\tau) + 1\right], \tag{45}$$

and

$$\Theta_R = \frac{AC}{6\sqrt{Bu_R}} \left[\tanh \sqrt{\left(\frac{u_R}{4B}\right)} (\zeta - u_R\tau) - 1\right]. \tag{46}$$

Substituting Eqs. (45) and (46) in Eq. (17), the trajectories of DKA solitons become

$$\begin{aligned} \zeta &= \epsilon^{1/2}(l_x x + l_z z - \lambda t) + \epsilon \frac{6C}{A}\sqrt{Bv_L} \\ &\left[\tanh\left[\left(\frac{v_L}{4B}\right)^{1/2}(\chi + v_L\tau)\right] + 1\right] + ..., \end{aligned} \tag{47}$$

$$\begin{aligned} \chi &= \epsilon^{1/2}(l_x z + l_z z + \lambda t) + \epsilon \frac{6C}{A}\sqrt{Bu_R} \\ &\left[\tanh\left[\left(\frac{u_R}{4B}\right)^{1/2}(\zeta - u_R\tau)\right] - 1\right] + ... \end{aligned} \tag{48}$$

Now, for getting the phase shift of the colliding soliton after collisions, it is assumed that the two solitons R and L before collision are distant from one another, *i.e.*, the two solitons R and L started their motion at $(\chi \to 0, \zeta \to -\infty)$ and $(\zeta \to 0, \chi \to +\infty)$, respectively, at the initial time $(t \to -\infty)$. After collision $(t \to +\infty)$, both solitons will change their positions, *i.e.*, R becomes at $(\chi \to 0, \zeta \to +\infty)$ and L becomes at $(\zeta \to 0, \chi \to -\infty)$. Taking all the above assumptions and following the same procedures as in Refs. [12, 13, 18], we finally get the phase shifts of the colliding solitons as

$$\Delta\Theta_R = 12\epsilon^{3/2}\frac{C}{A}\sqrt{Bv_L}, \tag{49}$$

$$\Delta\Theta_L = -12\epsilon^{3/2}\frac{C}{A}\sqrt{Bu_R}. \tag{50}$$

It is noted that the coefficient A is always positive and the coefficient B is always negative for physical validity of the soliton solution, hence the value of velocity $u_R(v_L)$ must be taken negative. The negative value of $u_R(v_L)$ means that $A < 0$ and $B > 0$, thus rarefactive sub-Alfénic solitons are formed.

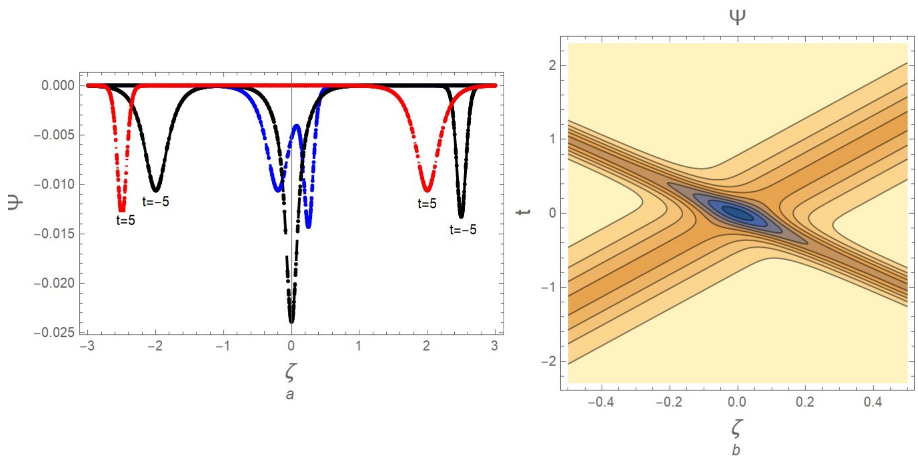

Fig. 1 – (a) The snapshot of the colliding process of unmodulated two DKA solitons is taken during the propagation, before collision $\tau = -5$, near collision $\tau = -0.5$, at collision $\tau = 0$, and after collision $\tau = 5$, and (b) Represents the contour plot for Fig. 1(a). Here, $\delta = 0.5$, $\epsilon = 0.1$, $\sigma = 0.2$, $l_z = 0.707$, $\alpha = 0.2$, $u = -0.5$, $v = -0.3$, and $\beta_d = 0.05$.

The scenario of counterpart propagating collision of rarefactive sub-kinetic Alfvén solitons having different amplitude (velocity) is illustrated in Figs. 1(a) and 1(b). It is displayed that the two solitons are at a distance from one another and as time pass on, the solitons approach one another. At time $\tau = 0$, they collided and form a merged nonlinear structure in the form of a single rarefactive soliton, which

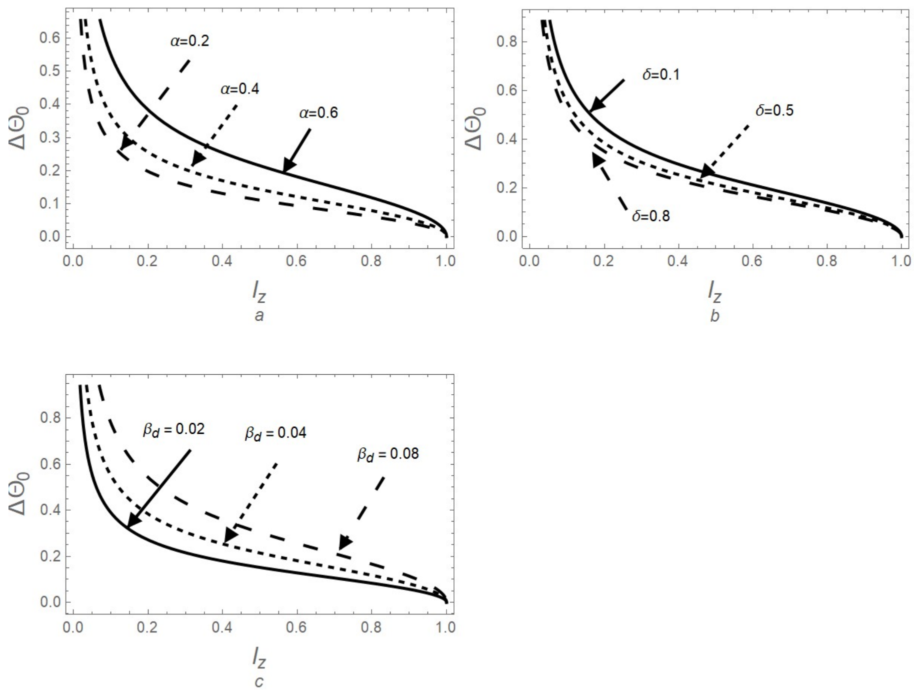

Fig. 2 – The phase shifts $\Delta\Theta_0$ due to the collision of two unmodulated solitons are plotted *versus* l_z for different values of (a) α with fixed values of $\delta = 0.5$, $\epsilon = 0.1$, $\sigma = 0.4$, $\beta_d = 0.04$, $u = v = -0.4$, (b) δ with fixed values of $\alpha = 0.2$, $\epsilon = 0.1$, $\sigma = 0.4$, $\beta_d = 0.04$, $u = v = -0.4$, and (c) β_d with fixed values of $\delta = 0.5$, $\epsilon = 0.1$, $\sigma = 0.4$, $u = v = -0.4$, $\alpha = 0.2$.

is depicted by the red curve. It is founded that the amplitude of the combined two solitons is nearly equal to the sum of individual amplitudes of two solitons. As the time goes on, they aside and maintain their structure but interchange their position as shown by the dashed red curves, respectively, at $\tau = 5$. It is noticed that the trajectories of the colliding solitons after collisions deviated from their trajectories before collisions. The effect of physical plasma parameters, obliqueness l_z, α, δ, and β_d on the phase shift is displayed in Fig. 2. It is noted that $\Delta\Theta_0$ decreases with the enhancement of l_z, α, and δ while β_d has an opposite effect on the phase shift, *i.e.*, the phase shift $\Delta\Theta_0$ increases with the increase of β_d. It is clear that the amplitude of the colliding soliton relates directly to its velocity. Therefore, the slowing soliton suffers more time in the collision region in comparison with the faster one. We can deduce that increasing the velocity u_R of the right soliton R involves grows in the amplitude Ψ_R, which leads to the enhancement of the phase shift of left soliton L and *vice versa*.

4. COLLISIONS OF MODULATED ENVELOPE SOLITONS

The KdV equation and its family (modified KdV, extended KdV, and so on) are known to be used for investigating the unmodulated structures that propagate with phase velocity. On the contrary, for describing the modulated wavepackages, the NLSE is used for this purpose. Thus, for studying the dark solitons collision in the framework of the KdV equation, an asymptotic method is used to convert the KdV Eq. (39) to a NLSE [52–59]. To do that, the solution of Eq. (39) can be written in the following modulated sinusoidal wave

$$\psi(\tau,\zeta) \equiv \Psi_{1\zeta} = \sum_{s=0}^{\infty}\sum_{l=-s}^{l=s} \psi_s^{(l)}\left(\tilde{\tau},\tilde{\zeta}\right) e^{il(k\zeta-\omega\tau)}, \tag{51}$$

where k and ω represent the fundamental wave number and frequency, respectively, of the nonlinear dust kinetic Alfvén wave. The independent variables can be stretched as follows

$$\tilde{\zeta} = \epsilon(\zeta - V_g\tau) \,\&\, \tilde{\tau} = \epsilon^2\tau,\ V_g \in \mathbb{R}. \tag{52}$$

Introducing Eqs. (51) and (52) into Eq. (39), we finally obtain

$$\begin{gathered}\left(\partial_{\tilde{\tau}} - \epsilon V_g\partial_\zeta + \varepsilon^2\partial_\tau\right)\left(\varepsilon\sum_{l=-1}^{1}\psi_1^{(l)}e^{il(k\zeta-\omega\tau)} + \epsilon^2\sum_{l=-2}^{2}\psi_2^{(l)}e^{il(k\zeta-\omega\tau)} + \ldots\right) + \\ A\left(\epsilon\sum_{l=-1}^{1}\psi_1^{(l)}e^{il(k\zeta-\omega\tau)} + \epsilon^2\sum_{l=-2}^{2}\psi_2^{(l)}e^{il(k\zeta-\omega\tau)} + \ldots\right)\times \\ \left(\partial_{\tilde{\zeta}} + \epsilon\partial_\zeta\right)\left(\epsilon\sum_{l=-1}^{1}\psi_1^{(l)}e^{il(k\zeta-\omega\tau)} + \epsilon^2\sum_{l=-2}^{2}\psi_2^{(l)}e^{il(k\zeta-\omega\tau)} + \ldots\right) + \\ B\left(\partial_{\tilde{\zeta}}^3 + 3\epsilon\partial_{(\tilde{\zeta}^2,\zeta)}^3 + 3\epsilon^2\partial_{(\tilde{\zeta},\zeta^2)}^3 + \epsilon^3\partial_\zeta^3\right) \\ \left(\epsilon\sum_{l=-1}^{1}\psi_1^{(l)}e^{il(k\zeta-\omega\tau)} + \epsilon^2\sum_{l=-2}^{2}\psi_2^{(l)}e^{il(k\zeta-\omega\tau)} + \ldots\right) = 0.\end{gathered} \tag{53}$$

Using relation (53), the following summarized orders are obtained for the same powers of ε,

- The coefficient of $\epsilon e^{i(k\zeta-\omega\tau)}$ (here $s = l = 1$) gives the linear dispersion relation for DKAWs *via* $\omega = -Bk^3$.

- The coefficient of $\epsilon^2 e^{i(k\zeta-\omega\tau)}$ (here $s = 2, l = 1$) gives the group velocity of the DKAWs *via* $V_g = -3Bk^2 \equiv \partial_k\omega$.

- The coefficient of $\epsilon^3 e^{i(k\zeta-\omega\tau)}$ (here $s = 3, l = 1$) with the last expressions yield the following NLSE

$$i\partial_{\tilde{\tau}}\Phi + P\partial_{\tilde{\zeta}}^2\Phi + Q|\Phi|^2\Phi = 0, \tag{54}$$

with

$$P = -3Bk \equiv \frac{1}{2}\partial_k^2\omega,$$
$$Q = \frac{A^2}{3Bk} \equiv \frac{-A^2}{P},$$

where $\Phi \equiv \psi_1^{(1)}$ and P and Q represent the coefficients of the dispersion and nonlinear terms, respectively.

It is shown that the modulated envelope structures in this case are always stable because the product PQ is always negative and its sign does not depend on the relevant physical parameters. Thus, in our case only dark solitons can propagate in the present plasma model. To pick up a lot of information (such as studying rogue waves and bright solitons) about the present plasma system, the NLSE for arbitrary wave number can be derived from the basic set of fluid equations using the derivative expansion method. Now, for studying the collisions between two envelope dark soliton, the extended PLK method is used for this purpose. Firstly, the solution of dark soliton of Eq. (53) is introduced in the following form [59, 78–81]

$$\Phi\left(\tilde{\zeta},\tilde{\tau}\right) = \Phi_0\left[1+G\right]e^{i\left(Q\Phi_0^2\tilde{\tau}+F\right)}, \tag{55}$$

where Φ_0 is an arbitrary constant, $G\left(\tilde{\zeta},\tilde{\tau}\right) \equiv G$, and $F\left(\tilde{\zeta},\tilde{\tau}\right) \equiv F$ denotes a phase function. Substituting solution (55) into Eq. (54), the following system of two equations is obtained

$$\partial_{\tilde{\tau}}G + P\left[2\partial_{\tilde{\zeta}}G\partial_{\tilde{\zeta}}F + (1+G)\,\partial_{\tilde{\zeta}}^2F\right] = 0, \tag{56}$$

and

$$-(1+G)\,\partial_{\tilde{\tau}}F + P\left[\partial_{\tilde{\zeta}}^2G - (1+G)\left(\partial_{\tilde{\zeta}}F\right)^2\right] + q\Phi_0^2\left[G\,(G+1)\,(G+2)\right] = 0. \tag{57}$$

Now let us assume that two envelope dark solitons propagate in opposite directions (say in right (S_R) and left (S_L) directions) in the present plasma model. After a certain period of time, the two solitons will meet and clash together, after that they move away from each other, leaving a single impact as a result of weakly nonlinearity and elastic collision, which is called the phase shift. For explanation of the mechanism/scenario of dark solitons collisions, the extended PLK method will be used to get the phase shift after collision. Accordingly, the independent variables quantities are stretched as follows [15, 81]:

$$\left.\begin{array}{l}\eta_R = \epsilon(\tilde{\zeta} - C_R\tau) + \epsilon^2\Pi_R^{(0)}(\eta_L) + ..., \\ \eta_L = \epsilon(\tilde{\zeta} + C_L\tau) + \epsilon^2\Pi_L^{(0)}(\eta_R) + ...\end{array}\right\} \tag{58}$$

Here, η_R and η_L give the trajectories of the right and left solitons, *i.e.*, R and L, respectively. The symbols C_R and C_L are for the velocities of the right and left solitons, respectively. The functions $P^{(0)}(\eta_L) \equiv P^{(0)}$ and $Q^{(0)}(\eta_R) \equiv Q^{(0)}$ express the change in the path of the two dark solitons after collision. The physical quantities in Eqs. (56) and (57) are expanded as:

$$\left.\begin{aligned} G &= \epsilon^2 G^{(0)} + \epsilon^4 G^{(1)} + ..., \\ F &= \epsilon F^{(0)} + \epsilon^3 F^{(1)} + ..., \\ C_{R,L} &= C_{R,L}^{(0)} + \epsilon^2 C_{R,L}^{(1)} + ..., \end{aligned}\right\} \tag{59}$$

where $C_R(\eta_R) \equiv C_R$ and $C_L(\eta_L) \equiv C_L$ for simplicity. With the inclusion of expansions (58) and (59) into Eqs. (56) and (57) we then follow the same procedure as in Refs. [15]. Here, we did not mention the details of the phase shift derivation to avoid distracting the reader from the basic idea. After tedious but straightforward calculations, the corresponding phase shifts between two envelope dark solitons are given by

$$\begin{aligned} \Delta\Pi_R &= \epsilon(\tilde{\zeta} - C_R\tau)\Big|_{(\eta_L,\eta_R)\to(0,-\infty)} - \epsilon(\tilde{\zeta} - C_R\tau)\Big|_{(\eta_L,\eta_R)\to(0,\infty)} \\ &= \varepsilon^2 \sqrt{\frac{H_L}{\Phi_0^2}\left(-\frac{P}{Q}\right)}, \end{aligned} \tag{60}$$

and

$$\begin{aligned} \Delta\Pi_L &= \epsilon(\tilde{\zeta} + C_L\tau)\Big|_{(\eta_L,\eta_R)\to(\infty,0)} - \epsilon(\tilde{\zeta} + C_L\tau)\Big|_{(\eta_L,\eta_R)\to(-\infty,0)} \\ &= -\varepsilon^2 \sqrt{\frac{H_R}{\Phi_0^2}\left(-\frac{P}{Q}\right)}. \end{aligned} \tag{61}$$

During our calculations the following relations are obtained: $C_R^{(0)} = C_L^{(0)} \equiv c = \sqrt{-2PQ\Phi_0^2}$, $C_R^{(1)} = -H_R c$, and $C_L^{(1)} = -H_L c$, where H_R and H_L represent the amplitude of right- and left-running soliton, *i.e.*, running soliton R and L, respectively. The collisions between the two envelope dark solitons are assumed to be weak and quasi-elastic. Therefore, the only effect that occurred as a result of the quasi-elastic collisions is a change in the trajectories after the colliding waves, which we express by phase shifts. Figure 3 demonstrates the scenario of dark solitons collision. The initial and individual amplitudes of the colliding right-moving dark soliton R and left-moving dark soliton L are, respectively, $H_R = 0.002$ and $H_L = 0.001$. It appears that both right- and left-dark-moving solitons remain unchanged after collision and the only change due to the collisions is the phase shift for both right and left dark solitons. The impact of δ, β_d, α, and l_z on the phase shift Π_R is elucidated in Fig. 4. It is clear that for growing both δ and l_z, the phase shift Π_R becomes smaller

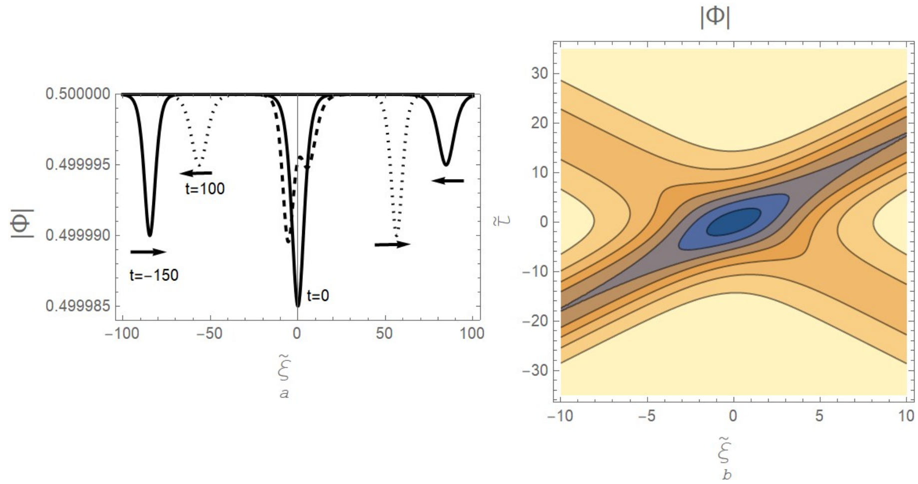

Fig. 3 – (a) The snapshot of the colliding process of modulated two DKA dark solitons is taken during the propagation, before collision $\tau = -150$, near collision $\tau = -10$, at collision $\tau = 0$, and after collision $\tau = 100$, and (b) The contour plot for the two dark solitons collision is plotted against $\left(\tilde{\zeta}, \tilde{\tau}\right)$. Here, $\delta = 0.1$, $\epsilon = 0.1$, $\sigma = 0.4$, $l_z = 0.7$, $\alpha = 0.4$, $\beta_d = 0.04$, $H_R = 0.002$, $H_R = 0.001$, $\Phi_0 = 0,5$, and $k = 0.5$.

as shown in Fig. 4(b). On the contrast, with respect to α and β_d, the phase shift Π_R increases with the enhancement of both α and β_d as shown in Figs. 4(a) and 4(c), respectively. Physically, enhancing or reducing the phase shift with increasing δ, α, β_d, and l_z can be illustrated according to the energy gain or loss during the collisions. It should be mentioned that the two phase shifts $\Delta\Pi_R$ and $\Delta\Pi_L$ are always contradictory, *i.e.*, if $\Delta\Pi_R > 0$ at appropriate value of the relevant physical parameters (here δ and l_z), we can find that $\Delta\Pi_L < 0$ at the same values of δ and l_z, and this is true according to the phase-conserving law [15]:

$$\frac{1}{\sqrt{H_R}}\Delta\Pi_R + \frac{1}{\sqrt{H_L}}\Delta\Pi_L = 0. \tag{62}$$

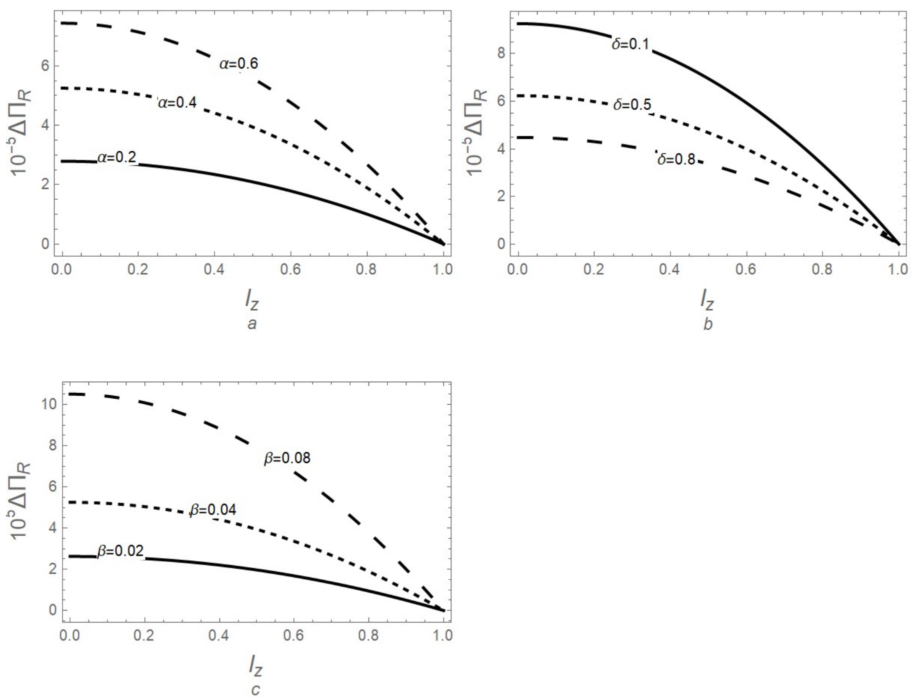

Fig. 4 – The phase shifts $\Delta\Pi_R$ due to the collision of two modulated dark solitons is plotted *versus* l_z for different values of (a) α with fixed values of $\delta = 0.5$, $\epsilon = 0.1$, $\sigma = 0.4$, $\beta_d = 0.04$, $H_R = 0.002$, $H_R = 0.001$, $\Phi_0 = 0,5$, $k = 0.5$, (b) δ with fixed values of $\epsilon = 0.1$, $\sigma = 0.4$, $\alpha = 0.4$, $\beta_d = 0.04$, $H_R = 0.002$, $H_R = 0.001$, $\Phi_0 = 05$, $k = 0.5$ and (c) β_d with fixed values of $\delta = 0.5$, $\epsilon = 0.1$, $\sigma = 0.4$, $\alpha = 0.4$, $H_R = 0.002$, $H_R = 0.001$, $\Phi_0 = 05$, $k = 0.5$.

5. CONCLUSIONS

In this study, the face-to-face (head-on) collision between two counterpart travelling dust kinetic Alfvén unmodulate and modulate solitons in inertial warm magnetized negative dust charged grains, inertialess Maxwell-Boltzmann distributed ions and electrons has been investigated. Our study is divided into two main parts. The first one is to study face-to-face unmodulated solitons collision. As is well-known that the unmodulated solitons set up due to the balance between the steeping and broadening phenomena. Now for studying the collisions between the unmodulated solitons, the basic equations of the physical problem are reduced to two KdV equations with two phase functions using the extended PLK method. It has been assumed during our study that the collisions between two unmodulated solitons are elastic and as a consequence the only impact attributable to the collision was the phase shift after collision. It is noted that for small amplitudes and quasi-elastic collisions, the maximum amplitude of the colliding solitons in the colliding area is approximately equal to the sum of individual amplitudes. The impact of physical parameters on the phase shifts is examined. The second part of our study is concerned with the study of collisions between two modulated dark solitons. The KdV equation is converted to the NLSE using a suitable transformation for describing the modulated envelope structures. After that and before embarking on a collision study, it is necessary to restrict the regions of stable and unstable modulated waves. It is proved in the literature [52–59] that the modulated waves become stable for the negative sign of the product of the coefficients of the nonlinear and dispersion terms, *i.e.*, $PQ < 0$, but for $PQ > 0$ the modulated waves become unstable. Depending on this condition it is found that when the KdV equation is converted to the NLSE, the unmodulated waves are always stable because PQ is always negative and the sign of this product does not depend on the value of the coefficients of the dispersion and nonlinear terms of the KdV equation. Now, we return to the main objective of the second part of the study, which is the study of collision between two dark collisions. To do that the PLK method is used to get the phase shift. The impact of the physical parameters on the phase shifts is reported. The present investigation is related to laboratory and space plasmas underpinning sub-dust Alfvénic soliton [42, 43].

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