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# ROLE OF OCTUPOLE DISTORTIONS OF THE FERMI SURFACE ON ELECTRIC ISOSCALAR COLLECTIVE MODES

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*Abstract.* The purpose of this work is to inquire on the continuum mechanical features of a system of strongly interacting fermions using the framework of Fermi liquids. The Landau equation for the Fermi distribution function is transformed into a set of coupled fluid-dynamical equations for multipole distortions of the Fermi surface up to $l = 3$. The set of fluid-dynamical equations for a one-component nuclear Fermi system is applied to the propagation of wave disturbances in infinite nuclear matter and electric isoscalar giant resonances in spherical nuclei.

*Key words*: Fermi system; nuclear matter; elasticity; giant resonance.

## 1. INTRODUCTORY REMARKS

Collective modes in electron systems [1] or atomic nuclei [2–8] can be viewed as elementary excitations of a Fermi liquid. In this framework, at very low temperatures, a collective mode may be pictured as an oscillation of the Fermi surface. In close analogy to a liquid drop, the distortion of the Fermi surface can be resolved *via* an expansion in a series of spherical surface harmonics [9]. Restricting this expansion to quadrupole distortions ($l_{\rm max} = 2$) of the Fermi surface, a closed set of equations is obtained for the macroscopic density and displacement variables that resembles the equations describing the elastic vibrations performed by an isotropic solid [3]. To date, apart of refs. [3, 4] that briefly discussed the dispersion relation for an infinite Fermi system when $l_{\rm max} = 3$, the effect of distortions of the Fermi surface of multipolarity higher than quadrupole for finite Fermi systems, such as nuclei, was not tackled in the literature. As pointed in the above cited references, the extension of $l_{\rm max}$ to infinity (Landau's zero sound regime) leads to a significant increase of the transverse sound velocity $c_T$. This fact has the consequence that longitudinal modes are more easy to excite.

On the other hand along the last six decades there have been many publications, especially in the literature dedicated to the study of giant resonances [10], that pointed out that due to the Fermi properties and the nature of nucleon-nucleon interaction, the nuclei display a high-frequency elastic reponse to external excitations [11–20].

Elastic properties of nuclear matter is a subject of renewed interest specially in relation to neutron star crusts studies [21] and in particular the nuclear pasta found in the inner crust [22]. The elastic properties of neutron star crusts are relevant for a wide range of electromagnetic and gravitational wave phenomena.

In a previous work [17] I reported calculations of the isoscalar giant dipole resonance (ISGDR) which displays a clear bimodal structure. Later on other groups published results that underscore the continuum mechanical (macroscopic) character of the ISGDR structure *e.g.* [23–25]. Apart of some fine structures of the dipole strength distribution, microscopic calculations confirm the gross bimodal structure of ISGDR. It is therefore of certain interest to investigate the effect of higher multipole distortions of the Fermi surface and to test the persistence of the ISGDR bimodal structure.

In Sec. 2 I derive the equations of motion for the macroscopic fields describing the distortion up to octupole contributions of the Fermi surface. This formalism is next applied to finite nuclei (Sec. 3) and nuclear matter (Sec. 4).

## 2. CONTINUUM MECHANICAL DESCRIPTION OF FERMI SURFACE DISTORTIONS (FSD)

The key role in describing the equilibrium and dynamics of a Fermi liquid is represented by the kinetic equation satisfied by the quasiparticle distribution function $n_p(\boldsymbol{r},t)$ in the absence of collisions and external forces [1]

$$\frac{\partial n_{\rm r}}{\partial t} + \boldsymbol{\nabla}_{\rm r} n_{\rm p} \cdot \boldsymbol{\nabla}_{\rm p} \varepsilon_{\rm p} - \boldsymbol{\nabla}_{\rm p} n_{\rm p} \cdot \boldsymbol{\nabla}_{\rm r} \varepsilon_{\rm p} = 0 \tag{1}$$

Collective modes in Fermi liquids within Landau's theory are investigated by allowing small amplitude fluctuations around the quasiparticle uniform equlibrium distribution function $n^0_{\rm p}$ [1]

$$n_p(\boldsymbol{r},t) = {n_{\rm p}}^0 + \delta n_{\rm p}(\boldsymbol{r},t) \tag{2}$$

where $\delta n_{\rm p}(\boldsymbol{r},t)$ is normalized such that its summation over "p" coincides with the difference between the number of particles $N$ in the fluctuated state and in the ground state, *i.e.*

$$N - N_0 = \sum_{\rm p} \delta n_{\rm p}(\boldsymbol{r},t) \tag{3}$$

The corresponding change in the quasiparticle energy is expressed in terms of the quasiparticle interaction

$$\delta\varepsilon_p(\boldsymbol{r},t) = \sum_{\rm p'} f_{\rm p,p'} \delta n_{\rm p}(\boldsymbol{r},t) \tag{4}$$

where $f_{\mathrm{p},\mathrm{p}'}$ is the second variational derivative of the total energy with respect to $n_{\mathrm{p}}$ and is interpreted as the interaction energy of the excited quasiparticles $\mathrm{p},\mathrm{p}'$ [1]. Above it was understood that the fluctuation $\delta n_p$ contains only long-wavelength components and is confined to the Fermi surface, *i.e.*

$$\delta n_p(\boldsymbol{r},t) = -\delta(\varepsilon_F - \varepsilon_p)\nu_{\theta_p,\phi_p}(\boldsymbol{r},t) \tag{5}$$

where $\nu_{\theta_p,\phi_p}$ is interpreted as the energy by which the zero sound wave shifts the quasiparticle distribution in the direction $\hat{\boldsymbol{p}} \equiv (\theta_p,\phi_p)$.

Substituing eq. (2) in eq. (1) and keeping only first-order terms in $\delta n_p$ the linearized form of the kinetic equation is obtained

$$\frac{\partial \delta n_{\mathrm{p}}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}_{\mathrm{r}}\delta n_p(\boldsymbol{r},t)\cdot\boldsymbol{v}_{\mathrm{p}} - \boldsymbol{\nabla}_p n_p^0 \cdot \sum_{\mathrm{p}'} f_{\mathrm{p},\mathrm{p}'}\cdot\boldsymbol{\nabla}_{\mathrm{p}}\delta n_{\mathrm{p}'}(\boldsymbol{r},t) = 0 \tag{6}$$

In view of the above mentioned assumption that the momentum vector $\boldsymbol{p}$ is strongly peaked in the vicinity of the Fermi surface, one can safely take $\boldsymbol{\nabla}_{\mathrm{p}} n_{\mathrm{p}}^0 = -\boldsymbol{v}_{\mathrm{p}}\delta(\varepsilon_{\mathrm{p}} - \varepsilon_{\mathrm{F}})$. In other words, the linearized kinetic equation describes solely quasiparticles of momentum $\boldsymbol{p}$.

Expanding in spherical harmonics the Fermi surface [9], the following series representations result for the quasi-particle energy shift and Fermi interaction,

$$\nu_{\theta_p,\phi_p}(\boldsymbol{r},t) = \sum_{l,m}\nu_{lm}(\boldsymbol{r},t)Y_{lm}(\theta_{\mathrm{p}},\phi_{\mathrm{p}}) \tag{7}$$

$$f_{\mathrm{p},\mathrm{p}'}(\boldsymbol{r},t) = \frac{1}{N(0)}\sum_{l,m}\frac{4\pi}{2l+1}F_l Y^*_{lm}(\theta_{\mathrm{p}'},\phi_{\mathrm{p}'})Y_{lm}(\theta_{\mathrm{p}},\phi_{\mathrm{p}}) \tag{8}$$

where the density of states at the Fermi surface,

$$N(0) = -\frac{1}{V}\sum_{\mathrm{p}}\frac{\partial n_{\mathrm{p}}^0}{\partial \varepsilon_{\mathrm{p}}}, \tag{9}$$

was introduced for convenience, and $F_l$ are the Landau-Fermi parameters [26]. Substituting the above two series in the linearized kinetic equation an infinite chain of equations for the multipole amplitudes $\nu_{lm}$ results

$$\begin{aligned}&\frac{\partial \nu_{lm}}{\partial t} + v_{\mathrm{F}}\sum_{l',m'}\left(1+\frac{F_{l'}}{2l'+1}\right)\\ &\times\int d\Omega_{\mathrm{p}} Y^*_{lm}(\theta_{\mathrm{p}},\phi_{\mathrm{p}})\sum_{L=l'\pm 1}\sqrt{\frac{L}{2l+1}}\mathbf{Y}^L_{lm}(\theta_{\mathrm{p}},\phi_{\mathrm{p}})\cdot\boldsymbol{\nabla}_{\mathrm{r}}\nu_{l'm'} = 0\end{aligned} \tag{10}$$

Let us consider perturbations of the distribution function that are periodic in space and time and are characterized by the wave-vector $\boldsymbol{q}$, *i.e.*

$$\nu_{l,m}(\boldsymbol{r},\boldsymbol{t}) = \nu_{lm}(\boldsymbol{q},\omega)e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)} \tag{11}$$

Thus, the following infinite set of coupled algebraic equations are obtained

$$\begin{aligned} \nu_{lm}(\boldsymbol{q},\omega) &= \frac{qv_{\mathrm{F}}}{\omega}\left[G_{l-1}\sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}\nu_{l-1m}(\boldsymbol{q},\omega)\right. \\ &+ \left. G_{l+1}\sqrt{\frac{(l+1-m)(l+1+m)}{(2l+1)(2l+3)}}\nu_{l+1m}(\boldsymbol{q},\omega)\right] \end{aligned} \tag{12}$$

where

$$G_l = 1+\frac{F_l}{2l+1} \tag{13}$$

Since the range of values of the Landau-Fermi parameters is constrained by the stability requirement [9], *i.e.*, the g.s. is a minimum, the $G_l$ coefficients are strictly positive. For example the zero-order harmonic coefficient $F_0$ is related to the compressibility of nuclear matter [2] and the condition $G_0 > 0$ insures the positiveness of this quantity. The condition $G_1 > 0$ signifies the positiveness of the effective mass. As shown bellow (see eq. (22)), $G_2 > 0$ in order to accomodate shear motions in the elastic Fermi liquid.

Following ref. [3] the following macroscopic variables that are associated with each multipole distortion of the Fermi surface are introduced:
density fluctuation (monopole)

$$\delta\rho = \nu_{00}, \tag{14}$$

current density fluctuation (dipole)

$$\delta\boldsymbol{\mathfrak{J}} = \frac{1}{M}\sum_{\mathrm{p}}\boldsymbol{p}\delta n_{\mathrm{p}}(\boldsymbol{r},t) = \frac{G_1 v_{\mathrm{F}}}{\sqrt{3}}\sum_m \nu_{1m}\boldsymbol{e}_\mu \tag{15}$$

displacement field (quadrupole)

$$\delta\boldsymbol{\mathfrak{s}} = \frac{\sqrt{5}}{2i\rho_0 q}\left(\nu_{20}\boldsymbol{e}_0 + \frac{2}{\sqrt{3}}(\nu_{21}\boldsymbol{e}_1 + \nu_{2-1}\boldsymbol{e}_{-1})\right) \tag{16}$$

and the ”octupole” vector field

$$\delta\boldsymbol{\mathfrak{O}} = \frac{3G_3}{2\sqrt{7}}\left(\nu_{30}\boldsymbol{e}_0 + \frac{2\sqrt{2}}{3}(\nu_{21}\boldsymbol{e}_1 + \nu_{2-1}\boldsymbol{e}_{-1})\right) \tag{17}$$

The above definitions were adopted with the intention to operate with analogous scalar and vector fields from continuum mechanics. Inserting the above definitions in the set of equations for the multipole amplitudes (10) and truncating for $l_{\max} = 3$

the following equations of motion for the four macroscopic fields are obtained:

$$\frac{\partial \delta\rho}{\partial t} + \boldsymbol{\nabla}\cdot\delta\mathfrak{J} = 0 \tag{18}$$

$$\frac{\partial \delta\mathfrak{J}}{\partial t} + \frac{1}{3}v_{\mathrm{F}}^2 G_0 G_1 \boldsymbol{\nabla}\delta\rho + \frac{1}{5}\rho_0 v_{\mathrm{F}}^2 G_1 G_2\left(\Delta\delta\mathfrak{s} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\delta\mathfrak{s})\right) = 0 \tag{19}$$

$$\rho_0\frac{\partial \delta\mathfrak{s}}{\partial t} + \delta\mathfrak{J} + v_{\mathrm{F}}\delta\mathfrak{O} = 0 \tag{20}$$

$$\frac{\partial \delta\mathfrak{O}}{\partial t} + \frac{8}{35}\rho_0 v_{\mathrm{F}} G_2\left(\Delta\delta\mathfrak{s} + \frac{1}{8}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\delta\mathfrak{s})\right) = 0 \tag{21}$$

For $l_{\max} = 2$ the last from the above set of equations is removed and the third term in the l.h.s of eq. (20) is not taken into account.

It was already pointed out in refs. [3] and [27] that if the Fermi's surface distortions are limited to multipole deformations not larger than $l = 2$ the equation of motion for the displacement field is completely analogous to the Lamé equation describing elastic waves [28]. In this approximation the Landau-Fermi parameters are related to the Lamé moduli of a linear isotropic elastic body:

$$\lambda = \frac{1}{3}\rho_0 v_{\mathrm{F}}^2 G_1\left(G_0 - \frac{2}{5}G_2\right), \quad \mu = \frac{1}{5}\rho_0 v_{\mathrm{F}}^2 G_1 G_2 \,. \tag{22}$$

Proceeding in an analogous manner to ref. [17] the scalar function $\mathcal{D}$ corresponding to the dilation-compression (volume specific deformations) and the vorticity vector $\boldsymbol{\omega}$, corresponding to rigid-body local rotations (Cauchy spin tensor), are introduced

$$\mathcal{D} = \boldsymbol{\nabla}\cdot\delta\mathfrak{s}, \quad \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}\times\delta\mathfrak{s} \,. \tag{23}$$

In terms of the components of the strain tensor $\mathfrak{E} \equiv (\boldsymbol{\nabla}\delta\mathfrak{s} + \delta\mathfrak{s}\boldsymbol{\nabla})/2$

$$\mathcal{D} = \varepsilon_{ii} \,. \tag{24}$$

This approach, consisting in the separation of the displacement field in a curless and a divergenceless component is frequently used in static and dynamic problems of the theory of elasticity [28].

In terms of these new variables the equations of motion (18)-(21) are written as

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \frac{1}{\rho_0}K\Delta\delta\rho + \frac{4}{3}\mu\Delta\mathcal{D} \tag{25}$$

$$\rho_0\frac{\partial^2 \mathcal{D}}{\partial t^2} = \frac{1}{\rho_0}K\Delta\delta\rho + \frac{4}{3}\mu''\Delta\mathcal{D} \tag{26}$$

$$\frac{\partial^2 \boldsymbol{\omega}}{\partial t^2} = \frac{\mu\,'}{\rho_0}\Delta\boldsymbol{\omega} \tag{27}$$

where

$$K = \lambda + \frac{2}{3}\mu \tag{28}$$

is the bulk (triaxial) compression modulus [28] which appears in association with the density fluctuations $\delta\rho$ as it comes visible from the above equations. The coefficients $\mu', \mu''$ are new shear moduli that take into account the octupole distortion of the Fermi surface,

$$\mu' = \mu\left(1 + \frac{8}{7}G_3\right), \quad \mu'' = \mu\left(1 + \frac{27}{28}G_3\right) . \tag{29}$$

The equation of motion for the displacement field is obtained by taking the time derivative of eq. (20) and substituting in the resulting expression eqs. (19) and (21),

$$\rho_0\frac{\partial^2\delta\mathfrak{s}}{\partial t^2} = \frac{1}{\rho_0}K\boldsymbol{\nabla}\delta\rho + \frac{1}{8}\left(\mu' + \frac{5}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\delta\mathfrak{s}) + \mu'\Delta\delta\mathfrak{s} , \tag{30}$$

This equation can be further be rewritten by introducing the force derived from the gradient of a stress tensor $\mathfrak{T}$ in the r.h.s, *i.e.*

$$\rho_0\frac{\partial^2\delta\mathfrak{s}}{\partial t^2} = \boldsymbol{\nabla} : \mathfrak{T} , \tag{31}$$

where ":" denotes the dyadic product [29] and the stress tensor assumes the dyadic form

$$\mathfrak{T} = \boldsymbol{e}_i\tau_{ij}\boldsymbol{e}_j , \tag{32}$$

where the summation is understood in the Einstein sense, *i.e.* after dummy labels. The components $\tau_{ij}$ of the stress tensor are then readily obtained

$$\tau_{ij} = \left[\frac{1}{\rho_0}K\delta\rho - \frac{7}{8}\left(\mu' - \frac{5}{21}\mu\right)\varepsilon_{kk}\right]\delta_{ij} + 2\mu'\varepsilon_{ij} \tag{33}$$

In the case $l_{\max} = 2$,

$$\delta\rho = \rho_0\varepsilon_{ii}$$

and accordingly the Lamé equation (30) and the stress-tensor assume the well-known forms from classical elasticity [28]

$$\rho_0\frac{\partial^2\delta\mathfrak{s}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\delta\mathfrak{s}) + \mu\Delta\delta\mathfrak{s} \tag{34}$$

$$\tau_{ij} = \lambda\varepsilon_{ii}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{35}$$

or in tensor form

$$\mathfrak{T} = \lambda(\boldsymbol{\nabla}\cdot\delta\mathfrak{s})\boldsymbol{I} + 2\mu\mathfrak{E} , \tag{36}$$

where $\boldsymbol{I} \equiv \boldsymbol{e}_i\delta_{ij}\boldsymbol{e}_j$ is the unit dyadic.

## 3. APPLICATIONS TO CHARGE-INDEPENDENT NUCLEAR SYSTEMS

### 3.1. FINITE SYSTEMS

For finite Fermi systems, the local Fermi momentum distribution in the surface region is anisotropic [30]. This anisotropy will induce couplings between the $m$-substates for a given multipolar state $l$ [4]. In what follows surface effects are neglected (for a qualitative discussion on nuclear surface effects on density fluctuations when taking into account multipole distortions of the Fermi surface see ref. [31]) and therefore the system of dynamical field equations deduced for infinite systems [3] is valid also for finite systems. The finite size of the Fermi system will come up in the boundary condition which is set for a given radius.

Considering monochromatic elastic waves of frequency $\Omega$ propagating in the finite Fermi system, the fluctuating parts of the density and the displacement field will undergo a harmonic variation in time

$$\delta\rho(\boldsymbol{r},t) = \delta\rho(\boldsymbol{r})e^{i\Omega t}, \quad \delta\mathfrak{s}(\boldsymbol{r},t) = \delta\mathfrak{s}(\boldsymbol{r})e^{i\Omega t}$$

In matrix form the compressibility and the vorticity are found to satisfy the scalar and vector Helmholtz equation respectively

$$\begin{pmatrix} c_S^2 & \frac{4}{3}{c_T}_1^2 & 0 \\ c_S^2 & \frac{4}{3}{c_T}_3^2 & 0 \\ 0 & 0 & {c_T}_2^2 \end{pmatrix} \left\{ \begin{array}{c} \Delta\delta\rho \\ \Delta\rho_0\mathcal{D} \\ \Delta\boldsymbol{\omega} \end{array} \right\} + \Omega^2 \left\{ \begin{array}{c} \delta\rho \\ \rho_0\mathcal{D} \\ \boldsymbol{\omega} \end{array} \right\} = 0 \tag{37}$$

where

$$c_S^2 = K/\rho_0, \quad {c_T}_{1,2,3}^2 = \{\mu, \mu', \mu''\}/\rho_0$$

and the sound velocity $c_S$ is related to the longitudinal phase velocity

$$c_L = \sqrt{(\lambda + 2\mu)/\rho_0},$$

according to the identity

$$c_S^2 = c_L^2 - \frac{4}{3}{c_T}_1^2 \, .$$

The first two scalar Helmholz equations describing density and dilation-compression vibrations are coupled whereas the last one, *i.e.* the vector Helmholz equation, describes shear oscillations.

Let me first concentrate on the equations satisfied by the scalars $\delta\rho, \mathcal{D}$ and write down the regular fundamental solutions of the scalar Helmholz equation corresponding to the eigenvalue of wavenumber $k_L$ [32]

$$\left\{ \begin{array}{c} \delta\rho \\ \mathcal{D} \end{array} \right\} (r,\theta,\phi) = \sum_{\lambda\mu} \left\{ \begin{array}{c} a_{\lambda\mu} \\ b_{\lambda\mu} \end{array} \right\} j_\lambda(k_L r) Y_{\lambda\mu}(\theta,\phi) \tag{38}$$

In the above expansion formula $j_\lambda$ stands for the spherical Bessel function.

In what follows I specialize to axial-symmetric dipole modes ($\lambda = 1, \mu = 0$). For that purpose the expression for the displacement field has to be corrected in order to account for the center-of-mass motion. The translational invariance is achieved by constraining the center-of-mass $\boldsymbol{R}_{\text{C.M.}}$ to be at rest for a finite spherical system of radius $R_0$ [15]:

$$\delta\boldsymbol{R}_{\text{C.M.}} = \frac{\int d\boldsymbol{r}\delta\mathfrak{s}(\boldsymbol{r})}{\int d\boldsymbol{r}} = 0 \tag{39}$$

Thence, the longitudinal component of the displacement field reads

$$\delta\mathfrak{s}_L(\boldsymbol{r}) = -\frac{1}{\sqrt{3}}b_{10}\left[\left(j_0(k_Lr) - \frac{3}{k_LR_0}j_1(k_LR_0)\right)\mathbf{Y}^0_{10}(\theta,\phi) + \sqrt{2}j_2(k_Lr)\mathbf{Y}^0_{12}(\theta,\phi)\right] \tag{40}$$

It is easy to check that by taking the divergence of the above vector the expression of the dilation, as given in eq. (38) for $\lambda = 1$ is obtained.

Substituting eq. (38) into eq. (37) a $2\times 2$ linear homogeneous set of equations for the amplitudes $a_{10}$ and $b_{10}$ is obtained. A non-trivial solution of this system results from the condition (dispersion relation)

$$k_L = \frac{\Omega}{\mathcal{C}(c_S^2, c_{T_1}^2, c_{T_3}^2)} \tag{41}$$

where

$$\mathcal{C}^{-2}(c_S^2, c_{T_1}^2, c_{T_3}^2) = \frac{1}{2}\frac{1}{c_{T_3}^2 - c_{T_1}^2}\left[\frac{3}{4} + \left(\frac{c_{T_3}}{c_{T_1}}\right)^2\right]\left[1 \pm \sqrt{1 - 3\frac{c_{T_3}^2 - c_{T_1}^2}{c_S^2}\left(\frac{4c_{T_1}^2}{3c_{T_1}^2 + 4c_{T_3}^2}\right)^2}\right] \tag{42}$$

Then, substituing back in one of the equations for amplitudes the ratio of the density to dilation amplitudes reads

$$\frac{a_{10}}{b_{10}} = \frac{4}{3}\rho_0\frac{c_{T_1}^2}{\mathcal{C}^2(c_S^2, c_{T_1}^2, c_{T_3}^2) - c_S^2} \tag{43}$$

Using the proportionality between the scalar fields $\delta\rho$ and $\mathcal{D}$, as given by the above ratio, the stress tensor (33) can be rewritten as

$$\tau_{ij} = \lambda'\mathcal{D}\delta_{ij} + 2\mu'\varepsilon_{ij} \tag{44}$$

where

$$\lambda' = \frac{1}{\rho_0}\frac{a_{10}}{b_{10}}\lambda - \mu G_3 \tag{45}$$

Projecting the stress tensor on the normal unit vector to the surface the elastic force acting on the nuclear sharp boundary is obtained

$$\boldsymbol{T}_n \equiv \boldsymbol{\mathfrak{T}}\cdot\boldsymbol{n} \tag{46}$$

After elementary calculations it can be easily shown that

$$\boldsymbol{T}_n = \lambda'\mathcal{D}\boldsymbol{n} + 2\mu'\left[(\boldsymbol{n}\cdot\boldsymbol{\nabla})\delta\mathfrak{s} + \boldsymbol{n}\times\boldsymbol{\omega}^{\mathrm{pol}}\right] \tag{47}$$

The vector field satisfying the last of the equations in (37) splits into a poloidal and a torsional solution (see Appendix of ref. [17]). Out of these two only the poloidal component is related to electric collective modes

$$\boldsymbol{\omega}^{\mathrm{pol}} \quad = \quad \sum_{\lambda\mu} c_{\lambda\mu} j_\lambda(k_T r)\mathbf{Y}^{\mu}_{\lambda\lambda}(\theta,\phi) \tag{48}$$

From the definition of the vorticity vector (23) the form of the axial-symmetric dipole transverse component of the displacement field constrained by translational invariance, reads

$$\delta\mathfrak{s}_T(\boldsymbol{r}) = -\frac{1}{\sqrt{3}}c_{10}\left[\left(j_0(k_T r) - \frac{3}{k_T R_0}j_1(k_T R_0)\right)\mathbf{Y}^0_{10}(\theta,\phi) - \frac{1}{\sqrt{2}}j_2(k_T r)\mathbf{Y}^0_{12}(\theta,\phi)\right] \tag{49}$$

The eigenmode analysis that allows the determination of frequencies consists in imposing boundary conditions at the surface of the system. On the sharp nuclear surface $\mathcal{S}$ the elastic moduli of the Fermi liquid assume the same values as in the interior bounded by this surface and consequently the elastic state of the body can be determined *via* the Neumann boundary condition. This is achieved by imposing the vanishing of the normal and tangential components of the above defined elastic force at the free surface

$$(\boldsymbol{n}\cdot\boldsymbol{T}_n)_{\mathcal{S}} \equiv \mathfrak{T}_{rr}|_{r=R_0} = 0, \qquad (\boldsymbol{n}\times\boldsymbol{T}_n)_{\mathcal{S}} \equiv (\boldsymbol{e}_\varphi\mathfrak{T}_{r\theta} - \boldsymbol{e}_\theta\mathfrak{T}_{r\varphi})|_{r=R_0} = 0 \tag{50}$$

where the stress vector $\boldsymbol{T}_n$, as indicated by eq. (47), is obtained by applying a first-order linear differential operator to $\delta\mathfrak{s}$. The above conditions provide two implicit equations for $k_L$ and $k_T$

$$b_{10}\left[(\lambda'+2\mu')k_L R_0 j_1(k_L R_0) - 4\mu' j_2(k_L R_0)\right] + 2\sqrt{2}c_{10}\mu' j_2(k_T R_0) = 0 \tag{51}$$

$$b_{10}2\sqrt{2}\mu' j_2(k_L R_0) + c_{10}\left(k_T R_0 j_1(k_T R_0) - 2j_2(k_T R_0)\right) = 0 \tag{52}$$

The boundary conditions provide also the ratio

$$r_n = \left|\frac{c^{(n)}_{10}}{a^{(n)}_{10}}\right| \tag{53}$$

a quantity indicating the admixture between the longitudinal (dilation-compression) and the transverse (vortical) field in a given eigenmode $n$. For the physical problem discussed in this paper only the first few eigenmodes (overtones) are relevant and are listed in Table 1.

*Table 1*

The first 4 overtones of the isocalar electric dipole mode provided by the eigenvalues of the boundary condition (50) and the vorticity/compressibility ratio for $^{208}$Pb

| Overtone ($n$) | $k_L^{(n)}$(fm$^{-1}$) | $k_T^{(n)}$(fm$^{-1}$) | $\hbar\Omega_n$ (MeV) | $\hbar\Omega_n/\hbar\omega_0$ | $r_n$ |
|---|---|---|---|---|---|
| 1 | 0.33 | 0.15 | 6.93 | 1.00 | 4.18 |
| 2 | 0.76 | 0.33 | 15.69 | 2.27 | 0.74 |
| 3 | 0.97 | 0.43 | 20.15 | 2.91 | 0.13 |
| 4 | 1.17 | 0.51 | 24.39 | 3.52 | 0.27 |

For the numerical investigation to be carried out below, I pick up from ref. [33] the Landau-Fermi liquid parameters for symmetric nuclear matter estimated using low momentum $N-N$ interactions of the CD-Bonn type. In the present investigation I assume a constant density inside the nucleus, equal to the saturation density of nuclear matter $\rho_0 = 0.16$ fm$^{-3}$ and nucleon radius $r_0 = 1.14$ fm. The following set of Landau parameters $F_0 = -0.476; F_1 = -0.335; F_2 = -0.238; F_3 = -0.101$ was taken from ref. [33] where the self-consistent solution of the Babu-Brown equations for low-momentum CD-Bonn potential is considered.

The application was made for the spherical nucleus $^{208}$Pb and comparing with the same case discussed in [17] for multipolarity $\leq 2$ a narrower spectrum is obtained with a new mode around $1\hbar\omega$ displaying a predominantly vortical structure of the displacement field. This result indicates that by including higher multipole deformations of the Fermi surface the isoscalar dipole response is enriched in the low-energy region. The appearence of eigenmodes in the $1\hbar\Omega$ and $3\hbar\Omega$ energy regions is consistent with microscopic calculations of the isoscalar dipole transition strenghts distribution in $^{208}$Pb as well the observation of the ISGDR bimodal structure reported by some experimental groups (see ref. [25] and references therein).

### 3.2. INFINITE SYSTEMS

It is worthwhile to complete this study with a discussion on elastic waves in an unbounded nuclear system. As mentioned in the introductory section, the problem of wave phenomena is a topic of certain relevance for astrophysical applications and several authors approached it by resorting to the concept of nuclear matter elasticity (see [21, 22] and references therein).

Let us consider a plane wave propagating in the direction of the wavenumber vector $\boldsymbol{k}$ and with circular frequency $\Omega$ in an unbounded Fermi system with octupole distortions of the Fermi surface included. The fluctuating scalar ($\delta\rho, \mathcal{D}$) and vector

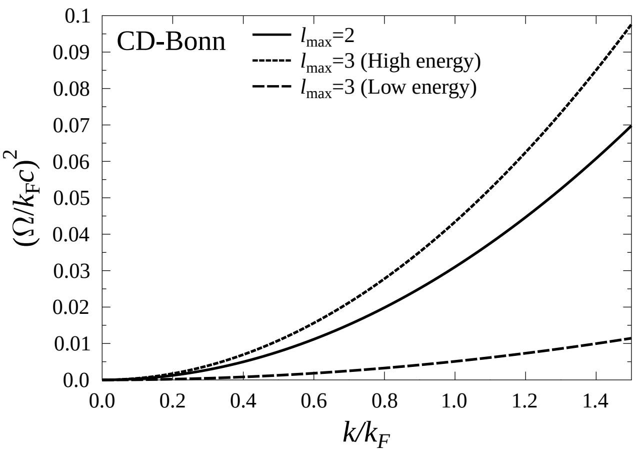


Fig. 1 – Dispersion law for the longitudinal elastic wave in nuclear matter for FSD with $l_{max} = 2$ (eq. (57), full line) and $l_{max} = 3$ (eq. (58)), low energy mode in short-dashed line and high energy mode in long-dashed lines calculated with Landau parameters extracted from the CD-Bonn potential.

($\boldsymbol{\omega}$) fields are represented by

$$\left\{ \begin{array}{c} \delta\rho \\ \mathcal{D} \\ \boldsymbol{\omega} \end{array} \right\} (\boldsymbol{r}, t) = \left\{ \begin{array}{c} a \\ b \\ \boldsymbol{c} \end{array} \right\} e^{i(\Omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \tag{54}$$

where $a, b$ and $\boldsymbol{c}$ are the corresponding space-independent amplitudes. Substituting the above *Ansatz* in the equations of motions (25-27) and employing the notations introduced in the above section, a dispersion relation is obtained, which next splits into a dispersion equation for longitudinal waves

$$\Omega^4 - k^2 \left( c_L^2 + \frac{9}{7} G_3 c_{T_1}^2 \right) \Omega^2 + \frac{9}{7} G_3 c_{T_1}^2 c_S^2 = 0 \tag{55}$$

and one for shear (rotational) waves

$$\Omega^2 = c_{T_2}^2 k^2 \tag{56}$$

For distortions of the Fermi surface with $l \leq 2$ the first dispersion relation reduce to

$$\Omega^2 = c_L^2 k^2 \tag{57}$$

When including octupole distortions of the Fermi surface a non-trivial effect is visible in the dispersion relation for longitudinal-dilation waves, *i.e.* two branches arise instead of one as in the case which include up to quadrupole distortions of the Fermi

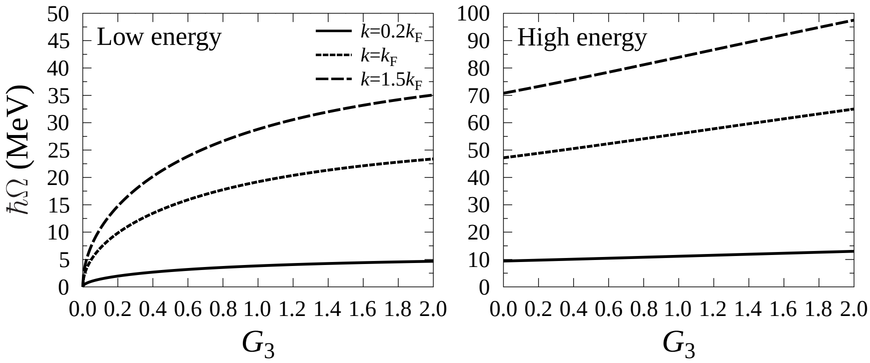


Fig. 2 – Evolution of the longitudinal elastic wave frequency in nuclear matter with the parameter $G_3$ for FSD with $l_{max} = 3$ and three values of the wavenumber : $k = 0.2k_F$ (full line), $k = k_F$ (short dashes) and $k = 1.5k_F$ (long dashes).

surface :

$$\Omega^2 = \frac{1}{2}\left(c_L^2 + \frac{9}{7}G_3 c_{T_1}^2\right)\left[1 \pm \sqrt{1 - \frac{36}{7}G_3\left(\frac{c_S c_{T_1}}{c_L^2 + \frac{9}{7}G_3 c_{T_1}^2}\right)^2}\,\right]k^2 \qquad (58)$$

The condition of existence of this second branch is:

$$c_L^2 + \frac{9}{7}G_3 c_{T_1}^2 > \sqrt{\frac{36}{7}G_3}\, c_S c_{T_1} \qquad (59)$$

In Fig. 1 the longitudinal dispersion law for FSD with multipolarity $l \leq 2$ is compared to the two branches resulting from the inclusion of the octupole distortions of the Fermi surface. I remark that for normal nuclear matter ($k = k_F$) the frequency of the $l \leq 3$ high energy component is approximately 1.15 larger than the frequency of the $l \leq 2$ longitudinal mode.

In Fig. 2 the dependence of the two modes (low energy on the left panel and high energy on the right panel) for FSD of multipolarities $l \leq 3$ on the Landau-Fermi parameter $G_3$ is displayed for three values of the wavenumber $k$. The first one ($k = 0.2k_F$) corresponds to dilute matter as encountered at the surface of finite nuclei, the second to normal nuclear matter ($k = k_F$) and the last one ($k = 1.5k_F$) to denser matter like the one produced in intermediate energy nuclear collisions or exists in the interior of neutron stars [21]. Since the CD-Bonn selection of the Landau-Fermi parameters provides $G_3 \approx 0.986$ one can safely conclude that the energies are only weakly sensitive to variations of $F_3$ that do not excceed the unity.

## 4. CONCLUSIONS

One of the main conclusions of the present study is that taking into account the octupole distortion of the Fermi surface it is possible to predict a response in the low-energy region which is consistent with sophisticated microscopic calculations made in the recent past as well indicated by very recent experimental data. It also strengthened the view that this type of nuclear collective excitations bears many resemblances with the elastic vibrations of a classical spherical body. A similar situation takes place in infinite systems where the inclusion of the octupole FSD gives rise to new ”branches“ of collective modes.

Since the role of octupole distortions of the Fermi surface is crucial in obtaining a low-energy component of the isoscalar dipole response, one should ask what can be expected if hexadecupole distortions ($l_{\max} = 4$) are further included in the calculations. The higher-order truncation in this case generates new vector fields which apparently have no analogue in continuum mechanics. Contrary to the quadrupole and octupole case, the vorticity in the hexadecupole case is no longer decoupled from the general set of equations of motion as preliminary calculations indicate [34].

The present method considered a single species of fermions and therefore it was justified as a tool to investigate isocalar modes where protons and neutrons move in phase. For isovector modes or neutron surface modes (such as the alleged dipole-soft mode) the Landau formalism should be extended to more than one species of fermions and derive the quasiparticle interaction for asymmetric nuclear matter [35]. In that case the second variational derivative of the total energy appearing in eq. (4) depends also on the relative spin- and isospin-orientations. In a forthcomming study I will continue the analysis carried out in the present paper and in one of my previous papers on proton-neutron fluid mixtures [36], to elastic two-component mixtures with material laws extracted from nuclear equations of state [37] and the Skyrme energy density functional [38].

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