

BRIGHT SOLITONS IN THE SPACE-SHIFTED \mathcal{PT} -SYMMETRIC NONLOCAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Under investigations in this paper are the bright solitons on the zero and periodic wave background in the space-shifted \mathcal{PT} -symmetric nonlocal nonlinear Schrödinger equation. These soliton solutions are constructed through the bilinear KP-hierarchy reduction method, and are given in terms of determinants. The collision dynamics of bright two-soliton solutions on the zero background are studied based on their asymptotic expressions. The bright four-soliton solutions can form bound state two-soliton pairs. The bright two-soliton solutions on the periodic wave background are also studied. Compared with the case of solitons on the zero background, the bright two-soliton solutions on the periodic wave background have completely different dynamics, even though the periodic waves vanish into the background.

Key words: \mathcal{PT} -symmetric nonlocal nonlinear Schrödinger equation, Bright solitons, KP-hierarchy reduction method.

1. INTRODUCTION

The nonlinear evolution equations (NLEEs) govern various physical phenomena in nature, thus they are of fundamental importance in physics and applied sciences [1–3]. Searching for exact solitary wave solutions of NLEEs in mathematical physics plays a crucial role in miscellaneous fields of science and engineering, such as nonlinear optics, fluid dynamics, plasma physics, and many others [4–7]. The nonlinear Schrödinger (NLS) equation and its solitary wave solitons are such typical examples, and they are applied in diverse physical settings [8–11]. Therefore, it is meaningful to study the soliton families of various NLEEs and the corresponding collision scenarios [12–26].

In the past years, the study on NLEEs with nonlocal parity–time (\mathcal{PT})-symmetry is a hot topic in the field of integrable systems. These nonlocal NLEEs were proposed since the seminal works of Bender and Boettcher [27, 28], in which they reported that although the Hamiltonian is non-Hermitian, it still can possess real energy eigenvalues if it obeys the \mathcal{PT} -symmetry. Due to the fact that the \mathcal{PT} -symmetry pro-

vides strong mathematical and physical insights to the underlying dynamical system, Ablowitz and Musslimani introduced in 2013 [29] the first nonlocal integrable system, namely, the \mathcal{PT} -symmetric nonlocal NLS equation:

$$\begin{aligned} iu_t(x, t) + u_{xx}(x, t) \pm 2u(x, t)V(x, t) &= 0, \\ V(x, t) &= u(x, t)u^*(-x, t). \end{aligned} \quad (1)$$

It contains a self-induced potential V that fulfils the \mathcal{PT} -symmetry condition $V^*(-x, t) = V(x, t)$. Equation (1) has been intensively studied, and its integrable properties as well as solution dynamics were considered from different points of view. Ablowitz and Musslimani developed the sophisticated inverse scattering transform scheme for this system to obtain soliton solutions with vanishing and non-vanishing boundary conditions [29–31]. Exact solitons, breathers, rational solitons, rogue wave solutions, and rogue waves on a background of periodic solitons in Eq. (1) have been investigated by different analytical methods [32–48], thereby exploring their dynamical features.

Very recently, Ablowitz and Musslimani [49] proposed the following space-shifted nonlocal NLS equation through a new nonlocal reduction of the AKNS hierarchy called space-shifted nonlocal reduction,

$$iu_t(x, t) + u_{xx}(x, t) + 2u^2(x, t)u^*(x_0 - x, t) = 0, \quad (2)$$

where the space-shifted factor x_0 is an arbitrary real parameter. When $x_0 = 0$, this space-shifted nonlocal NLS equation reduces back to the nonlocal NLS equation (1). Equation (2) has been solved by the inverse scattering transform and admits an infinite number of conservation laws [49]. Furthermore, the exact soliton, periodic soliton and rational rogue wave solutions of Eq. (2) have been studied by different analytical methods [50–57]. Very recently [58], we have considered the bright-dark solitons for two-component extensions of Eq. (2). Besides, we have studied the rogue waves and their patterns for Eq. (2) *via* the bilinear KP-hierarchy reduction method [57].

The focus of the present paper is on the dynamics of the multiple bright solitons and their interactions on the zero background and the periodic wave background in the space-shifted nonlocal NLS equation (2). We will first construct two families of multiple bright soliton solutions to Eq. (2) by the bilinear KP-hierarchy reduction method. The first family corresponds to solitons on the zero background while the other one are bright solitons on the periodic wave background. Then we will consider the dynamics of the obtained bright solitons on two different backgrounds. We found an interesting feature of these solitons. The intensities of the two bright solitons are equal in the case of the zero background. However, they are unequal in the case of the periodic wave background, even though the periodic waves on the background disappear into the background. This result was not reported before in the literature,

to the best of our knowledge.

The paper is organized as follows. In Sec. 2, we present the multiple soliton solutions on the zero and periodic wave background to nonlocal NLS equation (2) through two Theorems. In Sec. 3 we investigate in detail the dynamics of the bright solitons on both the zero and the periodic wave background in the nonlocal NLS equation (2). Section 4 presents the derivations of the obtained bright solutions in Sec. 2. The discussion of the obtained results and the conclusions are presented in Sec. 5.

2. THE BRIGHT SOLITON SOLUTIONS TO THE SPACE-SHIFTED \mathcal{PT} -SYMMETRIC NONLOCAL NLS EQUATION

In this Section we present two subclasses of single-pole bright soliton solutions to the space-shifted \mathcal{PT} -symmetric nonlocal NLS equation (2) on the zero and periodic waves background through two Theorems. For this purpose, we first bilinearize the nonlocal NLS equation (2) into the following forms

$$\begin{aligned} (D_x^2 + iD_t)g(x, t) \cdot f(x, t) &= 0, \\ D_x^2 f(x, t) \cdot f(x, t) &= 2cg(x, t)g^*(-x, t), \end{aligned} \quad (3)$$

through the dependent variable transformation

$$u = \frac{g(x, t)}{f(x, t)}, \quad (4)$$

where $c = \pm 1$, and D is the Hirota's bilinear differential operator [59], and the function $f(x, t)$ is subject to the following nonlocal symmetry and complex conjugated condition

$$f^*(x_0 - x, t) = cf(x, t). \quad (5)$$

Below, we present the multiple bright soliton solutions on the zero and periodic line waves backgrounds in terms of the determinants to the nonlocal NLS equation (2) through the following two Theorems, and the derivation of the bright soliton solutions is given in Sec. 4.

Theorem 1. *The space-shifted \mathcal{PT} -symmetric nonlocal NLS equation admits the following bright solitons on the zero background*

$$u = \frac{g(x, t)}{f(x, t)}, \quad (6)$$

where

$$f(x, t) = \begin{vmatrix} M_1 & M_2 \\ -\widehat{M}_2 & -\widehat{M}_1 \end{vmatrix}, g(x, t) = \begin{vmatrix} M_1 & M_2 & \Phi_1^T \\ -\widehat{M}_2 & -\widehat{M}_1 & \Phi_2^T \\ -\widehat{\Psi}_1 & -\widehat{\Psi}_2 & 0 \end{vmatrix}, \quad (7)$$

the wide hat ‘ $\widehat{}$ ’ represents the space-shifted \mathcal{PT} -symmetric condition $\widehat{h}(x, t) = h^*(x_0 - x, t)$, the elements $m_{s,j}^{(K)}$ of the matrix M_K are

$$m_{s,j}^{(1)} = \frac{1}{p_s + p_j^*} e^{\xi_s + \xi_j^*} - \frac{\alpha_s \alpha_j}{p_s + \bar{p}_j}, m_{s,j}^{(2)} = \frac{1}{p_s - p_j^*} e^{\xi_s + \tilde{\xi}_j^*} - \frac{\alpha_s \alpha_j^*}{p_s - p_j^*}, \quad (8)$$

$$\xi_s = p_s x + ip_s^2 t + \xi_{s,0}, \tilde{\xi}_j = -p_j(x - x_0) + ip_j^2 t + \xi_{j,0}^*,$$

for $s, j = 1, 2, \dots, N$, the superscript T represents the transpose, and $\Phi_1, \Phi_2, \bar{\Psi}_1$, and $\bar{\Psi}_2$ are row vectors defined by

$$\Phi_1 = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \Phi_2 = (e^{\tilde{\xi}_1}, e^{\tilde{\xi}_2}, \dots, e^{\tilde{\xi}_N}), \quad (9)$$

$$\bar{\Psi}_1 = (\alpha_1, \alpha_2, \dots, \alpha_N), \bar{\Psi}_2 = (\alpha_1^*, \alpha_2^*, \dots, \alpha_N^*).$$

Here $p_s, \xi_{s,0}, \alpha_s$ are freely complex parameters.

Theorem 2. *The space-shifted \mathcal{PT} -symmetric nonlocal NLS equation has the following bright solitons on a periodic wave background*

$$u = \frac{g(x, t)}{f(x, t)}, \quad (10)$$

where

$$f(x, t) = \begin{vmatrix} M_1 & M_2 & \Phi_{13}^T \\ -\widehat{M}_2 & -\widehat{M}_1 & \Phi_{23}^T \\ \Phi_{31} & \Phi_{32} & M_0 \end{vmatrix}, g(x, t) = \begin{vmatrix} M_1 & M_2 & \Phi_{13}^T & \Phi_1^T \\ -\widehat{M}_2 & -\widehat{M}_1 & \Phi_{23}^T & \widehat{\Phi}_2^T \\ \Phi_{31} & \Phi_{32} & M_0 & \Phi_0^T \\ -\bar{\Psi}_1 & -\bar{\Psi}_2 & -\bar{\Psi}_0 & 0 \end{vmatrix}, \quad (11)$$

$M_1, M_2, \Phi_1, \Phi_2, \Psi_1, \Phi_2$ are defined in Theorem 1, and $\Phi_{13}, \Phi_{23}, \Phi_{31}, \Phi_{32}$ are row vectors defined by

$$\Phi_{13} = (\Theta_1^{(1)}, \Theta_2^{(1)}, \dots, \Theta_N^{(1)}), \Phi_{23} = (\Theta_1^{(2)}, \Theta_2^{(2)}, \dots, \Theta_N^{(2)}),$$

$$\Phi_{31} = (\Delta_1^{(1)}, \Delta_2^{(1)}, \dots, \Delta_N^{(1)}), \Phi_{32} = (\Delta_1^{(2)}, \Delta_2^{(2)}, \dots, \Delta_N^{(2)}),$$

$$\Theta_s^{(1)} = \frac{1}{p_s + ip_0} e^{\xi_s + \widehat{\xi}_{2N+1}} - \frac{\alpha_s \alpha_{2N+1}^*}{p_s + ip_0},$$

$$\Theta_s^{(2)} = -\frac{1}{p_s - ip_0} e^{\tilde{\xi}_s^* + \widehat{\xi}_{2N+1}} + \frac{\alpha_s^* \alpha_{2N+1}^*}{p_s - ip_0}, \quad (12)$$

$$\Delta_j^{(1)} = \frac{1}{ip_0 + p_j^*} e^{\xi_{2N+1} + \xi_j^*} - \frac{\alpha_{2N+1} \alpha_j}{p_s + p_j^*},$$

$$\Delta_j^{(2)} = \frac{1}{ip_0 - p_j^*} e^{\xi_{2N+1} + \tilde{\xi}_j} - \frac{\alpha_{2N+1} \alpha_j^*}{ip_0 - p_j^*},$$

$$\xi_{2N+1} = ip_0 x + ip_0^2 t + \xi_{2N+1,0},$$

and M_0, Φ_0 are given by

$$M_0 = \frac{e^{2(ip_0x + \xi_{2N+1,0})}}{2ip_0} - \frac{|\alpha_{2N+1}|^2}{2ip_0}, \Phi_0 = e^{\xi_{2N+1}}, \Psi_0 = \alpha_{2N+1}^*. \quad (13)$$

Here $\xi_{2N+1,0} = \xi_0 - \frac{i}{2}p_0x_0$, and ξ_0, p_0 are freely real parameters.

3. THE DYNAMICS OF BRIGHT SOLITONS ON THE ZERO AND PERIODIC WAVE BACKGROUND

In this Section we study in detail the dynamics of the bright solitons on the zero and periodic wave backgrounds.

3.1. DYNAMICS OF BRIGHT SOLITONS ON THE ZERO BACKGROUND

The bright two-soliton solutions of nonlocal NLS equation (2) can be derived from Theorem 1 by putting $N = 1$ in Eqs. (6)–(9), and their determinant forms are explicitly written as

$$u = \frac{g}{f}, \quad (14)$$

where

$$f = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,1}^{(2)} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,1}^{(1)} \end{vmatrix}, g = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,1}^{(2)} & e^{\xi_1} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,1}^{(1)} & e^{\widetilde{\xi}_1} \\ -\alpha_1 & -\alpha_1^* & 0 \end{vmatrix}, \quad (15)$$

$m_{1,1}^{(1)} = \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} - \frac{\alpha_1^2}{p_1 + p_1^*}, m_{1,1}^{(2)} = \frac{e^{\xi_1 + \widetilde{\xi}_1}}{p_1 - p_1^*} - \frac{|\alpha_1|^2}{p_1 - p_1^*}$, and $\xi_1 = p_1x + ip_1^2t + \xi_{1,0}, \widetilde{\xi}_1 = -p_1(x - x_0) + ip_1^2t + \xi_{1,0}^*$. These bright two-soliton solutions are characterized by three free complex parameters $p_1, \xi_{1,0}, \alpha_1$. Besides, they undergo head-on collisions and their velocities are $2p_{1I}$ and $-2p_{1I}$. Here and in the following, R and I appearing in the subscript represent the real and imaginary parts of a given parameter or a function, respectively.

To study the collision of the bright two-soliton solutions, we derive the asymptotic properties of these waveforms. For this purpose, we assume $p_{1,R} > 0, p_{1,I} > 0$, and for convenience we define the right-moving soliton along the line $\zeta_1 = x - 2p_{1I}t \approx 0$ as soliton 1 and the left-moving soliton along the line $\zeta_2 = x + 2p_{1I}t \approx 0$ as soliton 2, then the above bright two-soliton solutions have the following asymptotic forms:

(a) Before collision ($t \rightarrow -\infty$):

Soliton 1 ($\zeta_1 \approx 0, \zeta_2 \rightarrow +\infty$):

$$u_1^- \simeq X_1 e^{-\lambda_1 + i\xi_{1I}} \operatorname{sech}(\xi_{1R} + \lambda_1); \quad (16)$$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \rightarrow +\infty$):

$$u_2^- \simeq X_2 e^{-\lambda_2 + i\xi_2 t} \operatorname{sech}(\zeta_2 + \lambda_2); \quad (17)$$

(b) After collision ($t \rightarrow +\infty$):

Soliton 1 ($\zeta_1 \approx 0, \zeta_2 \rightarrow -\infty$):

$$u_1^+ \simeq \bar{X}_1 e^{-\bar{\lambda}_1 + i\xi_1 t} \operatorname{sech}(\xi_{1R} + \bar{\lambda}_1); \quad (18)$$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \rightarrow -\infty$):

$$u_2^+ \simeq \bar{X}_2 e^{-\bar{\lambda}_2 + i\xi_2 t} \operatorname{sech}(\zeta_2 + \bar{\lambda}_2). \quad (19)$$

The auxiliary functions and quantities in the above expressions are defined by

$$\begin{aligned} X_1 &= -\frac{p_1(p_1 + p_1^*)}{\alpha_1(p_1 - p_1^*)}, e^{2\lambda_1} = \frac{4p_1 p_1^*}{\alpha_1^2(p_1 - p_1^*)^2}, \\ X_2 &= \frac{p_1(p_1 + p_1^*)}{\alpha_1^*(p_1 - p_1^*)}, e^{2\lambda_2} = \frac{4p_1 p_1^*}{\alpha_1^{*2}(p_1 - p_1^*)^2}, \\ \bar{X}_1 &= -\frac{p_1^2 - p_1^{*2}}{4\alpha_1 p_1}, e^{2\hat{\lambda}_1} = \frac{(p_1 - p_1^*)^2}{4\alpha_1^2 |p_1|^2}, \\ \bar{X}_2 &= \frac{p_1^2 - p_1^{*2}}{4\alpha_1^* p_1}, e^{2\hat{\lambda}_2} = \frac{(p_1 - p_1^*)^2}{4\alpha_1^{*2} |p_1|^2}. \end{aligned} \quad (20)$$

From the above asymptotic analyses we can obtain $|u_j^+(\xi_{jR})| = |u_j^-(\xi_{jR} - \lambda_j + \bar{\lambda}_j)|$ for $j = 1, 2$, which can further indicate that the amplitudes and velocities of the solitons remain unaltered after collision except for finite shifts, hence the two-soliton waveforms exhibit elastic collisions. Besides, the amplitudes of the two solitons are equal and they are

$$A = \left| \frac{p_{1R}}{\alpha_{1I}} \right| |\alpha_1|.$$

Here we have to note that the phase shift factor x_0 does not affect the amplitudes and velocities of these bright two solitons. Figure 1 shows the bright two solitons for $p_1 = 1 + i, \xi_{1,0} = 0, \alpha_1 = 2i$ and different values of x_0 , and we can see that only the phases are different. The amplitudes of these two solitons are equal to 1.

Taking $N = 2$ in Eqs. (6)–(9), the bright four-soliton solutions to the nonlocal NLS equation (2) can be generated

$$u = \frac{g}{f}, \quad (21)$$

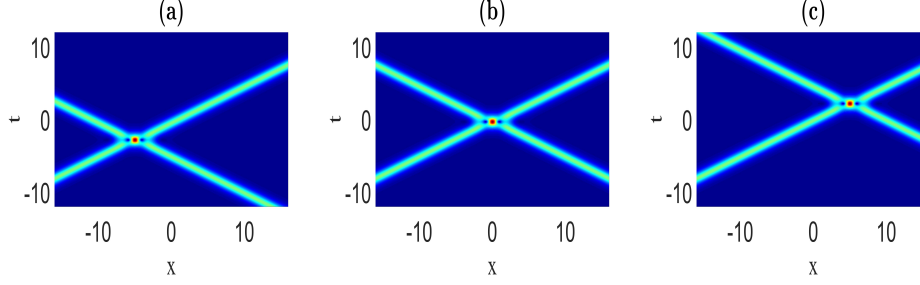


Fig. 1 – (Colour online) The bright two soliton solution (14) with parameters $p_1 = 1 + i$, $\xi_{1,0} = 0$, $\alpha_1 = 2i$ and different values of x_0 : (a) $x_0 = -10$; (b) $x_0 = 0$; (c) $x_0 = 10$.

where

$$f = \begin{pmatrix} m_{1,1}^{(1)} & m_{1,2}^{(1)} & m_{1,1}^{(2)} & m_{1,2}^{(2)} \\ m_{2,1}^{(1)} & m_{2,2}^{(1)} & m_{2,1}^{(2)} & m_{2,2}^{(2)} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,2}^{(2)} & -\widehat{m}_{1,1}^{(1)} & -\widehat{m}_{1,2}^{(1)} \\ -\widehat{m}_{2,1}^{(2)} & -\widehat{m}_{2,2}^{(2)} & -\widehat{m}_{2,1}^{(1)} & -\widehat{m}_{2,2}^{(1)} \end{pmatrix}, g = \begin{pmatrix} m_{1,1}^{(1)} & m_{1,2}^{(1)} & m_{1,1}^{(2)} & m_{1,2}^{(2)} & e^{\xi_1} \\ m_{2,1}^{(1)} & m_{2,2}^{(1)} & m_{2,1}^{(2)} & m_{2,2}^{(2)} & e^{\xi_2} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,2}^{(2)} & -\widehat{m}_{1,1}^{(1)} & -\widehat{m}_{1,2}^{(1)} & \widetilde{e}^{\xi_1} \\ -\widehat{m}_{2,1}^{(2)} & -\widehat{m}_{2,2}^{(2)} & -\widehat{m}_{2,1}^{(1)} & -\widehat{m}_{2,2}^{(1)} & \widetilde{e}^{\xi_2} \\ -\alpha_1 & -\alpha_2 & -\alpha_1^* & -\alpha_2 & 0 \end{pmatrix}, \quad (22)$$

and the matrix elements $m_{s,j}^{(1)}, m_{s,j}^{(2)}$ are given by Eq. (8). There are six free parameters in the bright four-soliton solutions: $p_j, \xi_{j,0}, \alpha_j$ for $j = 1, 2$. The imaginary parts of the parameters p_1, p_2 determine the states of the four bright solitons and can be classified into two types: (i) two bound state two-soliton solutions when $p_{1I} = p_{2I}$; and (ii) four crossed solitons when $p_{1I} \neq p_{2I}$. Figures 2(a), 2(b) show these two types of bright four solitons, which are at different states. Particularly, when $p_2 \approx p_1$ and $p_{1I} \approx 0$, the bright four solitons feature waveforms similar to the fourth-order pole solitons, see Fig. 2(c).

3.2. DYNAMICS OF BRIGHT SOLITONS ON THE PERIODIC WAVE BACKGROUND

In this Section we focus on the dynamics of bright solitons on the periodic wave background. To this end, we first consider a periodic solution of the nonlocal NLS equation (2), which provides a periodic wave background in solutions (10). By setting $N = 0$ in Eqs. (10)–(13), the periodic solution is expressed as

$$u = 2ip_0\alpha_1^* \frac{e^{ip_0(x - \frac{x_0}{2}) - ip_0^2 t + \xi_0}}{e^{2ip_0(x - \frac{x_0}{2}) + 2\xi_0} - |\alpha_1|^2}. \quad (23)$$

The maximum amplitude of the periodic waves is $A_p = \left| \frac{2\alpha_1 p_0 e^{\xi_0}}{e^{2\xi_0} - |\alpha_1|^2} \right|$, and they are only periodic along x and their period is $\frac{\pi}{p_0}$. Particularly, $|u| \rightarrow 0$ as $|\alpha_1| \rightarrow \infty$, namely,

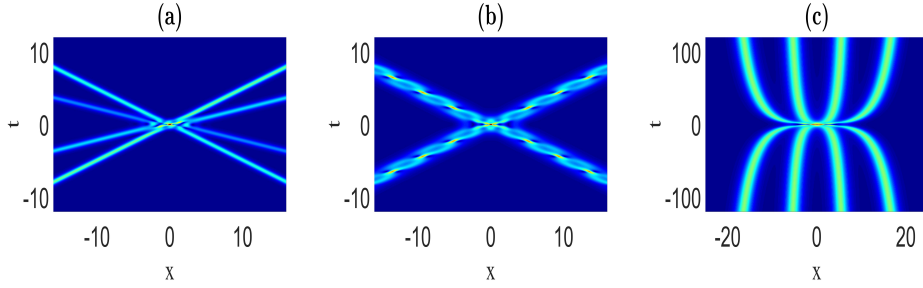


Fig. 2 – (Colour online) Three different types of bright four solitons with $x_0 = 0$: (a) the four crossed solitons with parameters $p_1 = 1 + i, \xi_{1,0} = 0, \alpha_1 = 2i$ and different values of $p_1 = 2 + i, p_2 = 1 + 2i, \xi_{1,0} = 0, \xi_{2,0} = 0, \alpha_1 = i, \alpha_2 = i$; (b) two bound state two-solitons with parameters $p_1 = 2 + i, p_2 = 1 + i, \xi_{1,0} = 0, \xi_{2,0} = 0, \alpha_1 = i, \alpha_2 = i$; (c) the four solitons featuring wave structures similar to the fourth-order pole solitons with $p_1 = 1 + 10^{-6}i, p_2 = \frac{10001}{10000} + 10^{-6}i, \xi_{1,0} = 0, \xi_{2,0} = 0, \alpha_1 = i, \alpha_2 = i$.

the periodic wave solution will vanish into the zero background. The dynamics of the periodic wave solutions and their intensity plots with different values of α_1 are shown in Fig. 3. From the panel (b) of Fig. 3 we can directly observe that the periodic wave amplitudes will be changed for different values of α_1 .

Then we consider the two solitons on the background of above periodic waves. By taking $N = 1$ in Eqs. (10)–(13), the solutions are explicitly expressed as

$$u = \frac{g}{f}, \quad (24)$$

where

$$f = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,1}^{(2)} & \Theta_1^{(1)} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,1}^{(1)} & \Theta_1^{(2)} \\ \Delta_1^{(1)} & \Delta_1^{(2)} & M_0 \end{vmatrix}, g = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,1}^{(2)} & \Theta_1^{(1)} & e^{\xi_1} \\ -\widehat{m}_{1,1}^{(2)} & -\widehat{m}_{1,1}^{(1)} & \Theta_1^{(2)} & e^{\widetilde{\xi}_1} \\ \Delta_1^{(1)} & \Delta_1^{(2)} & M_0 & e^{\xi_3} \\ -\alpha_1 & -\alpha_1^* & -\alpha_3^* & 0 \end{vmatrix}, \quad (25)$$

$$m_{1,1}^{(1)} = \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} - \frac{\alpha_1^2}{p_1 + p_1^*}, m_{1,1}^{(2)} = \frac{e^{\xi_1 + \widetilde{\xi}_1^*}}{p_1 - p_1^*} - \frac{|\alpha_1|^2}{p_1 - p_1^*}, \xi_1 = p_1 x + ip_1^2 t + \xi_{1,0}, \widetilde{\xi}_1 = -p_1(x - x_0) + ip_1^2 t + \xi_{1,0}^*, \Theta_1^{(1)} = \frac{1}{p_1 + ip_0} e^{\xi_1 + \xi_3} - \frac{\alpha_s \alpha_3^*}{p_1 + ip_0}, \Theta_1^{(2)} = -\frac{1}{p_1 - ip_0} e^{\widetilde{\xi}_1 + \xi_3} + \frac{\alpha_s^* \alpha_3^*}{p_1 - ip_0}, \Delta_2^{(1)} = \frac{1}{ip_0 + p_1^*} e^{\xi_3 + \xi_1^*} - \frac{\alpha_3 \alpha_1}{p_1 + p_1^*}, \Delta_1^{(2)} = \frac{1}{ip_0 - p_1^*} e^{\xi_3 + \widetilde{\xi}_1} - \frac{\alpha_3 \alpha_1^*}{ip_0 - p_1^*}, \xi_3 = ip_0 x + ip_0^2 t + \xi_{3,0}, \xi_{3,0} = \xi_0 - \frac{i}{2} p_0 x_0.$$

The dynamics of bright two solitons on the periodic wave background are characterized by two sets of parameters: $p_1, \xi_{1,0}, \alpha_1$ and p_0, ξ_0, α_3 . The former one mainly determines the dynamics of the bright two solitons while the later one mainly affects the dynamics of the periodic wave background. The effects of parameter α_3 is equivalent to that of parameter α_1 in solution (23). By enlarging the value of α_3 , the

periodic wave would degenerate into the background, and the background would become a constant one. Figure 4 depicts the solutions (24) with different values of α_3 . In panel (a) of Fig. 4, we can see that the two solitons become periodic solitary waves due to the interaction with periodic waves. In panel (b) of Fig. 4, the periodic waves have vanished into the constant background, and the two solitons propagate similarly as on the zero background previously discussed. However, the intensities of the two solitons are unequal. This result is different from that in the case of the zero background, where the intensities of the two solitons are equal. That change may come from the energy transformation between the two bright solitons and periodic waves even though the two solitons seem to vanish into the background.

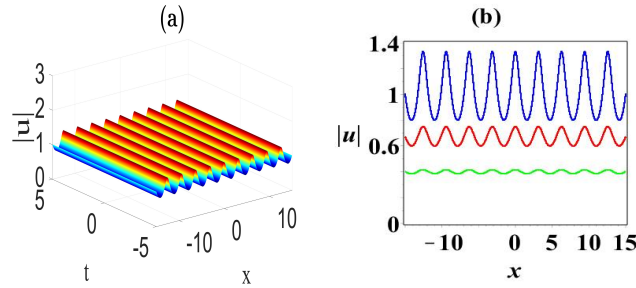


Fig. 3 – (Colour online) (a) The periodic wave solution (23) to the nonlocal NLS equation (2) with parameters $x_0 = 0, p_0 = 1, \xi_0 = 0, \alpha_1 = 2$. (b) The intensity plot of the periodic wave solution (23) at time $t = 0$ with parameters $x_0 = 0, p_0 = 1$, and $\alpha_1 = 2$ (blue line), $\alpha_1 = 3$ (red line), and $\alpha_1 = 5$ (green line).

4. DERIVATION OF THE GENERAL HIGHER-ORDER ROGUE WAVE SOLUTIONS

In this Section we will present the derivations of the multiple bright soliton solutions to nonlocal NLS equation (2) in Theorem 1 and Theorem 2 *via* the KP hierarchy reduction method [59–62], which is a powerful method to construct various types of solitary waves to integrable systems [63–71].

Referring to Sato theory [60–62], the bilinear equations in the KP hierarchy

$$\begin{aligned} (D_{x_1}^2 - D_{x_2})\tau_1 \cdot \tau_0 &= 0, \\ D_{x_1} D_y \tau_0 \cdot \tau_0 &= -2\tau_1 \tau_{-1}, \end{aligned} \quad (26)$$

admit the following tau functions expressed in Gram-type determinant form

$$\tau_0 = |M|, \tau_1(\ell) = \begin{vmatrix} M & \Phi^T \\ -\bar{\Psi} & 0 \end{vmatrix}, \tau_{-1} = \begin{vmatrix} M & \Psi^T \\ -\bar{\Phi} & 0 \end{vmatrix}, \quad (27)$$

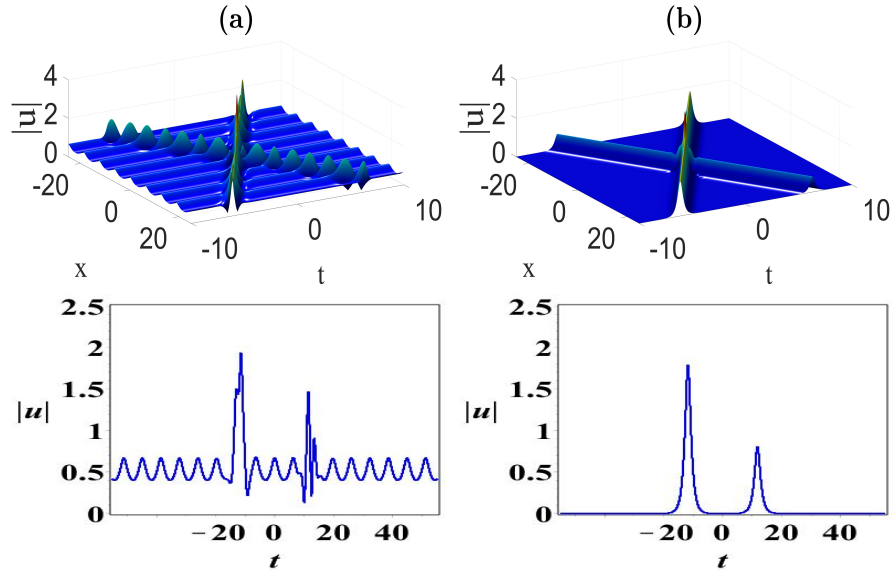


Fig. 4 – (Colour online) The dynamics of two-soliton solutions (24) on a background of periodic waves with parameters $p_1 = 1 - 2i$, $\xi_{1,0} = 0$, $\alpha_1 = 1 + i$, $p_0 = -\frac{1}{2}i$, $\xi_0 = 0$, $x_0 = 1$ and different values of α_3 : (a) $\alpha_3 = 2$; (b) $\alpha_3 = 25$. The bottom row are the intensity plots of top row at time $t = 4$.

where the elements of the matrix M are

$$m_{s,j} = \frac{1}{p_s + \bar{p}_j} e^{\xi_s + \bar{\xi}_j} + \frac{1}{q_s + \bar{q}_j} e^{\eta_s + \bar{\eta}_j}, \quad (28)$$

with

$$\begin{aligned} \xi_s &= p_s x_1 + p_s^2 x_2 + \xi_{s,0}, \bar{\xi}_j = \bar{p}_j x_1 + \bar{p}_j^2 x_2 + \bar{\xi}_{j,0}, \\ \eta_j &= q_j y + \eta_{s,0}, \bar{\eta}_j = \bar{q}_j y + \bar{\eta}_{s,0}, \end{aligned} \quad (29)$$

for $1 \leq s, j \leq K$, K is a positive integer, and the superscript T represents the transpose, Φ , Ψ , $\bar{\Phi}$, and $\bar{\Psi}$ are row vectors defined by

$$\begin{aligned} \Phi &= (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_K}), \Psi = (e^{\eta_1}, e^{\eta_2}, \dots, e^{\eta_K}), \\ \bar{\Phi} &= (e^{\bar{\xi}_1}, e^{\bar{\xi}_2}, \dots, e^{\bar{\xi}_K}), \bar{\Psi} = (e^{\bar{\eta}_1}, e^{\bar{\eta}_2}, \dots, e^{\bar{\eta}_K}). \end{aligned} \quad (30)$$

In what follows, we implement the dimension procedure for function τ_0 , which is rewritten as

$$\tau_0 = \prod_{s=1}^K e^{\xi_s + \bar{\xi}_s} |m'_{s,j}|, \quad (31)$$

where

$$m'_{s,j} = \frac{1}{p_s + \bar{p}_j} + \frac{1}{q_s + \bar{q}_j} e^{\eta_s - \xi_s + \bar{\eta}_j - \bar{\xi}_j}. \quad (32)$$

Since

$$(\partial_{x_1} - \partial_y) m'_{s,j} = \left(1 + \frac{p_s + \bar{p}_j}{q_s + \bar{q}_j}\right) e^{\eta_s - \xi_s + \bar{\eta}_j - \bar{\xi}_j}, \quad (33)$$

thus, we can obtain

$$\partial_{x_1} m'_{s,j} = \sum_{\ell=1}^M \delta_\ell \partial_{y_\ell} m'_{s,j}, \quad (34)$$

if the parameters p_s, q_s satisfy the following parametric constraints

$$q_s = -p_s, \bar{q}_j = -q_j, \quad (35)$$

for $s, j = 1, 2, \dots, K$. That further implies the relation

$$\partial_{x_1} \tau_0 = \partial_y \tau_0. \quad (36)$$

Under the above dimension reduction, the bilinear equations in (26) give rise to the following bilinear equation

$$D_{x_1}^2 \tau_0 \cdot \tau_0 = -2\tau_1 \tau_{-1}. \quad (37)$$

Therefore, the bilinear equations in Eq. (26) become to the following bilinear equations

$$\begin{aligned} (D_{x_1}^2 - D_{x_2}) \tau_1 \cdot \tau_0 &= 0, \\ D_{x_1}^2 \tau_0 \cdot \tau_0 &= -2\tau_1 \tau_{-1}. \end{aligned} \quad (38)$$

Then y is a dummy variable, thus we take $y = 0$.

Under the variable transformations $x_1 = x, x_2 = it$, and if the tau functions satisfy the nonlocal symmetry reduction

$$\tau_0^*(x_0 - x, t) = \tau_0(x, t), \tau_1^*(x_0 - x, t) = -\tau_{-1}(x, t), \quad (39)$$

or

$$\tau_0^*(x_0 - x, t) = -\tau_0(x, t), \tau_1^*(x_0 - x, t) = \tau_{-1}(x, t), \quad (40)$$

the above tau function reduces to the solutions of bilinear equations (3) by taking $f = \tau_0, g = \tau_1, g^*(x_0 - x, t) = c\tau_{-1}$, then bright soliton solutions in Theorems 1, 2 to the nonlocal NLS equations (2) can be derived.

In the following step, we will realize the nonlocal symmetry reduction in Eqs. (39), (40) with the variable transformations $x_1 = x, x_2 = it$. To this end, we first consider $K = 2N \times 2N$ matrices for the tau functions τ_0, τ_1 , and τ_{-1} defined in Eq. (27), and we take the following parametric conditions:

$$\begin{aligned} p_{N+s} &= -p_s, \bar{p}_{N+s} = -\bar{p}_s, \bar{p}_s = p_s^*, \bar{\xi}_{s,0} = \xi_{s,0}^*, \bar{\eta}_{j,0} = \eta_{j,0}, \\ \xi_{N+s,0} &= \xi_{s,0}^* + p_s x_0, \bar{\xi}_{N+s,0} = \xi_{s,0} + p_s^* x_0, \\ \eta_{N+s,0} &= \eta_{j,0}^*, \bar{\eta}_{N+s,0} = \bar{\eta}_{j,0}^*, \end{aligned} \quad (41)$$

for $s = 1, 2, \dots, N$. Under these parameter constraints, we obtain

$$\begin{aligned} (\xi_s + \bar{\xi}_j)(x, t) &= (p_s + p_j^*)x + i(p_s^2 - p_j^{*2})t + \xi_{s,0} + \xi_{j,0}^*, \\ (\xi_{N+s} + \bar{\xi}_{N+j})(x, t) &= (p_s + p_j^*)(x_0 - x) + i(p_s^{*2} - p_j^2)t + \xi_{s,0}^* + \xi_{j,0}, \\ \bar{\xi}_s(x, t) &= p_s^*x - ip_s^*t + \xi_{s,0}, \\ \xi_{N+s}(x, t) &= p_s(x_0 - x) + ip_s^2t + \xi_{s,0}^*, \end{aligned} \quad (42)$$

which implies

$$\begin{aligned} (\xi_s + \bar{\xi}_j)^*(x_0 - x, t) &= (\xi_{N+s} + \bar{\xi}_{N+j})(x, t), \\ (\xi_s + \bar{\xi}_{N+j})^*(x_0 - x, t) &= (\xi_j + \bar{\xi}_{N+s})(x, t), \\ (\xi_{N+s} + \bar{\xi}_j)^*(x_0 - x, t) &= (\xi_{N+j} + \bar{\xi}_s)(x, t), \\ (\xi_{N+j} + \bar{\xi}_{N+s})^*(x_0 - x, t) &= (\xi_s + \bar{\xi}_j)(x, t). \end{aligned} \quad (43)$$

Therefore

$$\begin{aligned} m_{K+s, K+j}^*(-x, t) &= -m_{j, s}(x, t), m_{K+s, j}^*(-x, t) = -m_{K+j, s}(x, t), \\ m_{s, K+j}^*(x_0 - x, t) &= -m_{j, K+s}(x, t), m_{s, j}^*(x_0 - x, t) = -m_{K+j, K+s}(x, t), \end{aligned} \quad (44)$$

which give rise to

$$\tau_0^*(x_0 - x, t) = \tau_0(x, t), \tau_1^*(x_0 - x, t) = -\tau_{-1}(x, t), \quad (45)$$

namely, the nonlocal symmetry reduction in Eq. (39) has been realized. Hence the bilinear equations (38) reduces to the bilinear equations (3) of the nonlocal NLS equation (2) with $f = \tau_0, g = \tau_1, g^*(x_0 - x, t) = -\tau_{-1}$ and $c = 1$, and the tau functions defined in (27) with variable transformations $x_1 = x, x_2 = it$ and under the parametric constraints in (35) and (41) reduce to the tau functions of the bilinear equations of nonlocal NLS equations (3), which can generate bright soliton solutions to the nonlocal NLS equation (2) with zero boundary condition. By defining

$$m_{s, j} = m_{s, j}^{(1)}, m_{N+s, j} = m_{s, j}^{(2)}, \Phi_1 = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \alpha_s = e^{\eta_{s,0}}, \quad (46)$$

and

$$m_{N+s, j} = -\widehat{m}_{s, j}^{(2)}, m_{N+s, N+j} = -\widehat{m}_{s, j}^{(1)}, \Phi_2 = (e^{\widetilde{\xi}_1}, e^{\widetilde{\xi}_2}, \dots, e^{\widetilde{\xi}_N}), \alpha_s^* = e^{\eta_{N+s,0}}, \quad (47)$$

then we obtain the multiple bright soliton solutions to the nonlocal NLS equation in Theorem 1. That finishes the proof of Theorem 1.

Finally, we perform the nonlocal symmetry reduction in Eq. (40). For this purpose, we consider $K = (2N + 1) \times (2N + 1)$ matrices for the tau functions τ_0, τ_1 , and τ_{-1} defined in (27), and take some parameters meeting the constraint condition in Eq. (41) and the remaining parameters following the parametric constraint condition

$$\begin{aligned}
p_{2N+1} &= \bar{p}_{2N+1} = ip_0, \bar{\xi}_{2N+1,0} = \xi_{2N+1,0}, \\
\xi_{2N+1,0} &= \xi_0 - \frac{i}{2}p_0x_0, \bar{\eta}_{2N+1,0} = \eta_{2N+1,0}^*.
\end{aligned} \tag{48}$$

Therefore the matrix elements $m_{s,j}$ still hold for the identities in Eq. (44) for $s, j = 1, 2, \dots, N$, and the remaining elements $m_{2N+1,j}, m_{2N+1,N+j}, m_{N+s,2N+1}, m_{s,2N+1}$ and $m_{2N+1,2N+1}$ meet the following relations:

$$\begin{aligned}
m_{2N+1,j}^*(x_0 - x, t) &= -m_{N+j,2N+1}(x, t), \\
m_{s,2N+1}^*(x_0 - x, t) &= -m_{2N+1,N+s}(x, t), \\
m_{2N+1,2N+1}^*(x_0 - x, t) &= -m_{2N+1,2N+1}(x, t),
\end{aligned} \tag{49}$$

similarly in Eq. (45), we obtain

$$\tau_0^*(x_0 - x, t) = -\tau_0(x, t), \tau_1^*(x_0 - x, t) = \tau_{-1}(x, t). \tag{50}$$

By further defining $\Theta_s^{(1)} = m_{s,2N+1}, \Theta_s^{(2)} = m_{N+s,2N+1}, \Delta_j^{(1)} = m_{2N+1,j}, \Delta_j^{(2)} = m_{2N+1,N+j}, M_0 = m_{2N+1,2N+1}, \Phi_0 = e^{\xi_{2N+1}}, \Psi_0 = \alpha_{2N+1}^*$, we obtain the solutions to the nonlocal NLS equation (2) in Theorem 2. That completes the proof of Theorem 2.

5. CONCLUSION AND DISCUSSION

We have studied in this paper the multiple bright solitons on the zero and periodic wave background for the space-shifted \mathcal{PT} -symmetric nonlocal NLS equation (2). These bright soliton solutions on two different types of background, expressed through $2N \times 2N$ and $(2N + 1) \times (2N + 1)$ determinants, respectively, have been constructed by using the bilinear KP-hierarchy reduction method. We first studied the collisions of the bright solitons on the zero background, and have discovered two interesting characteristics: (i) the two solitons possess equal intensities; and (ii) they only exhibit elastic collisions. Although the bright two-soliton solutions cannot form bound states, but the bound state two-soliton pairs have been found in the four-soliton solutions. Another types of bright four-soliton solutions have also been found, which feature waveforms similar to the fourth-order pole bright solitons. Then we have investigated the bright two-soliton solutions on the periodic wave background. In contrast with the case of solitons on the zero background, the two-soliton solutions on the periodic wave background have two completely different characteristics: (i) their intensities can be unequal (the intensities are equal on the zero background); and (ii) they can form periodic solitary waves along both x and t axes.

There are two extensions of the present work:

- The multiple bright pole solitons in the nonlocal NLS equation (2).

- The multiple bright solitons in the multi-component extension of Eq. (2), namely,

$$iu_t^{(j)} + u_{xx}^{(j)} + u^{(j)} \left(\sum_{j=1}^{\hat{N}} \delta_j u^{(j)} u^{(j)*}(x_0 - x, t) \right) = 0, \quad j = 1, 2, \dots, \hat{N}, \quad (51)$$

The above two extensions will be reported in detail elsewhere.

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