THE USE OF LAPLACE TRANSFORM IN THE INVERSE PROBLEM FROM BOUND STATES

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Abstract. In a previous work dedicated to non-relativistic quantum mechanics in the three-dimensional space with spherical symmetry, we have shown the possibility of reconstructing the ground state density from its moments in the case of bound states. The method is based on relationships connecting these moments to the lower energy of each state of angular momentum $\ell$. The ground state density yields the ground state wave function, which allows us to calculate the equivalent local potential by inverting the Schrödinger equation. This calculus uses the Fourier transform of the ground state density $\rho(r)$. The aim of this work is to check how to improve the results by replacing the Fourier transform by the Laplace transform of the quantity $r^2\rho(r)$. The method is first presented. Then examples and results are compared to those obtained previously.

Key words: Quantum mechanics, inverse problem, Laplace transform, moments of the ground state density.

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1. INTRODUCTION

The inverse problem, i.e. the determination of the potential from data, has a long history. A large number of works has been devoted to this subject. Among them the most frequently quoted are those of Bargmann [1], Marchenko [2], Jost and Kohn [3], and Gel’fand and Levitan [4]. More details can be found in textbooks of Newton [5], Chadan and Sabatier [6], and Grosse and Martin [7].

A large amount of work were dedicated to scattering data coupled or not with bound states and the usual procedure is to postulate parametrized expressions to the potential and search for the parameters that best fit the data. The case of bound states
has been less addressed. It began to arouse the interest with the discovery of heavy quark systems [8, 9].

Considering bound state spectra, our method is based on the generalized Bertlmann-Martin inequalities [10–14]. These recurrence relationships connect the even moments of the ground state density to the energy differences between the "yrast" levels ($1\ell$ for different values of $\ell$) and the ground state ($1s$, $\ell = 0$). Extensive tests on known potentials showed the moments to closely coincide with the exact values. The moments being known, the ground state density is reconstructed by means of its Fourier transform, which can be expressed by an infinite series of the ground state moments. The method was explained in previous papers [14, 15] showing in detail how to reach an equivalent local potential from energy spectra. The uniqueness condition of the solution is specified and the limitations of the method have been identified. The most important point is the necessary (sufficient) number of moments, and thus of energies, to recover the potential with a reasonable accuracy. Thus, it has been found that in simple cases, 15-20 moments are enough to get 1% accuracy or better [14]. The method was extended to the case of complex potentials and applications to hadronic atoms were done. Qualitative results on nuclear potential was deduced [15]. More recently, we can cite our work in the $D = 1$ dimensional space [16] and on the special case of complex potentials with real spectra [17, 18].

In this work we propose to reconstruct the ground state wave function by the use of the Laplace transform $L(q)$ of the square of the reduced density $\chi^2(r) = r^2 \rho(r)$. The key point is to verify if an equivalent accuracy can be reached with fewer moments than in case of Fourier transform. As we shall see, the fundamental difference in using Laplace transform is the necessity to determine moments of both parities, the Fourier transform requiring only the even ones. Because the odd moments are not directly involved in the Bertlmann-Martin inequalities, an appropriate interpolation scheme has to be found.

The paper is organized as follows: the method is recalled in Sec. 2. Results are presented and discussed in Sec. 3. Conclusions are drawn in Sec. 4.

2. THE METHOD

The starting point of our method is the generalized Bertlmann-Martin inequalities. Assuming spherical symmetry in the three-dimensional Schrödinger equation for two particles interacting via a local potential (or a particle moving in a central potential), one can use the operator $Q_{\ell,0}(\vec{r}) = r^\ell Y_{\ell,0}(\theta, \phi)$ to derive the sum rules linear in energy:

$$
(E_{1\ell} - E_{1s}) \sum_n | < 0|Q_{\ell,0}|n\ell> |^2 \leq \sum_n (E_{n\ell} - E_{1s}) | < 0|Q_{\ell,0}|n\ell> |^2.
$$

(1)
Here, $n_\ell$ denotes the $n$-th level of angular momentum $\ell$; the number of nodes of the wave function is given by $n - 1$. $E_{1\ell}$ is the lowest eigenvalue of angular momentum $\ell$. The use of the closure relation and the sum rule estimate lead to [10, 11]:

$$<r^{2\ell}>_{1s} \leq \frac{\hbar^2}{2m}(2\ell + 1)\frac{<r^{2\ell-2}>_{1s}}{(E_{1\ell} - E_{1s})}$$

In these expressions,

$$<r^k>_{1s} = \int_0^\infty |R_{1s}(r)|^2 r^{k+2} dr = \int_0^\infty \rho_{1s}(r)r^{k+2} dr$$

is the $k$-th moment of the ground state density; use is made here of $\hbar = 2m = 1$. $R_{1s}(r)$ is radial wave function. The relation (2) gives only an upper limit for the different moments of the ground state density. A correction factor can be introduced to saturate these inequalities, i.e. to convert them to equalities. Its exact expression is unknown, a priori. An exact form was proposed by Bertlmann and Martin for $k = 2$ [10], and generalized later to all orders [11]. Relation (2) is then multiplied by:

$$f(\ell) = 1 - \frac{\ell}{2(\ell + 1)} \left[ \frac{E_{(\ell+1)s} + E_{1s} - 2E_{1\ell}}{E_{(\ell+1)s} - E_{1s}} \right]^2.$$ (4)

This expression is exact for the harmonic oscillator and Coulombic potential and can be used as a first guess for other potentials and improved by an iterative procedure.

The ground state density moments are then obtained by the corrected Bertlmann-Martin relationships and used to write the Fourier transform of the ground state density as:

$$F(q) = \int \rho(\vec{r}) e^{-i\vec{q} \cdot \vec{r}} d\vec{r} = \sum_{n=0}^\infty (-)^n \frac{<r^{2n}>}{(2n+1)!} q^{2n}.$$ (5)

This formal series expansion is valid only inside the convergence radius, which is finite. This implies that $F(q)$ is known only up to a maximum value of the momentum transfer $q_{max}$. To extend this maximum value, we use the Padé approximants that allows us to write $F(q)$ as a rational function $P(N,D)$ of two polynomials of degree $N$ and $D$, respectively [19]. The choice of the fraction is not unique. However, by taking the average of a sufficient number of fractions we can reduce the uncertainties. Actually, for small values of $q$ all fractions give the same results for $F(q)$. However, for large values of $q$ corresponding to the small values of $r$ in the ground state density, we note a dispersion of the different fractions. This issue has been widely discussed in previous work [12, 14, 15].

The ground state density is obtained by the inverse Fourier transform of $F(q)$

$$\rho(\vec{r}) = \int F(q) e^{i\vec{q} \cdot \vec{r}} d\vec{q} = \int P(N,D) e^{i\vec{q} \cdot \vec{r}} d\vec{q}.$$ (6)
The ground state wave function, which is positive definite, is calculated by taking the square root of the ground state density. The potential is then obtained by inverting the Schrödinger equation for the reduced wave function:

\[ V(r) = E_{1s} + \frac{\chi''_{1s}(r)}{2\chi_{1s}(r)}; \quad \chi_{1s}(r) = rR_{1s}(r). \]  

The ground state density and the potential are in general calculated numerically. The derived potential is unique if the number of the known moments and therefore the number of energy states is infinite. In practice, tests have shown a good accuracy to be reachable with a finite number of available moments. On the other hand, if the number of experimentally measured bound states is limited, the potential is approached with uncertainties, which has to be estimated. See, for example, our discussion of this point in the case of hadronic atoms in the previous work [15].

The aim of the present work is to check if substituting the Fourier transform by the Laplace transform can improve the method, and the consequences on the number of available moments.

Consider the quantity \( r^2 \rho(r) \). Its Laplace transform can be written as:

\[ L(q) = \int r^2 \rho(r) e^{-qr} dr = \sum_{n=0}^{\infty} (-)^n \frac{<r^n>}{n!} q^n. \]  

Whereas the even moments are taken from the generalized Bertlmann-Martin inequalities, an appropriate interpolation has to be used to get the odd moments. The next step of the procedure is identical to the one given above. Padé approximants allows us to write \( L(q) \) as continuous fractions \( P(N,D) \). The inverse Laplace transform is given by [20]:

\[ f(r) = r^2 \rho(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{qr} L(q) dq. \]  

Here, \( c \) is an arbitrary real number greater than the real parts of all the singularities [21]. There are some simple functions for which the inverse Laplace transform can be obtained analytically. In most cases, a numerical integration is required. Many methods have been proposed based on series expansion [20]. Care must be taken in this respect, because the different procedures are not effective in all cases. Comparison of results obtained in different ways limits errors, and underlines wrong values.

### 3. EXAMPLES

Studying the possible improvements brought by the Laplace transform, in the following examples we consider cases for which the exact ground state density is known. Note that in the whole calculations, we have used the exact ground state
moments. We recall that our main goal here is to test the use of Laplace transform. Then, it is not necessary to use those calculated with the Bertlmann-Martin inequalities, shown to be close to the exacts ones [12]. The results of a numerical example are given in the end of the Section.

3.1. COULOMB POTENTIAL

We consider the Coulombic potential:

$$V(r) = -\frac{1}{r}.$$  

The ground state radial part of wave function is given by:

$$R_{1s}(r) = 2e^{-r},$$  (10)

and the exact moments of the ground state density have the expression:

$$\langle r^k \rangle_{1s} = \frac{(k + 2)!}{2k + 1}.$$  (11)

The energy spectrum is expressed by:

$$E_{n\ell} = -\frac{1}{2(n + \ell + 1)^2} : n = 0, 1, 2, \ldots; \ell = 0, 1, 2, \ldots$$  (12)

The first values of the ground state moments are given in Table 1. The calculation of Laplace transform leads to a unique fraction, namely $P(0,3)$. All other fractions with $D > 3$ are all equal to $P(0,3)$. On the other hand, $P(0,3)$ is equal to the exact
Laplace transform:
\[ L(q) = \frac{8}{(2+q)^3}. \]  

(13)

For comparison, when using the Fourier transform, it is the \( P(0, 4) \) Padé approximant that gives the exact result. The case of Coulombic potential is particular because it is possible to construct the exact ground state density from a small number of its moments. The two methods are effective and the improvement made by using the Laplace transform is not significant. The ground state density obtained by inverse Laplace transform is exact and the ground state wave function deduced is exact too (relation (10)). Both ways lead to the exact potential.

3.2. KRATZER POTENTIAL

The Kratzer molecular potential is written as [22]:
\[ V(r) = -2B\left(\frac{a}{r} - \frac{a^2}{2r^2}\right), \]  

(14)

where \( B \) and \( a \) are real constants such as \( V(a) = -B \). The reduced ground state wave function takes the form:
\[ \chi_{1s}(r) = rR_{1s}(r) = cr^2e^{-2r}F(0, 4, 2r), \]  

(15)

where \( c \) is the normalization constant and \( F(0, 4, 2r) \) designates the confluent hypergeometric function [23].

For our calculations we consider \( B = 3/8 \) and \( a = 1 \). Under these conditions, the energy spectrum is given by:
\[ E_{n,\ell} = -\frac{9}{32}\left[n + \frac{1}{2} + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + \frac{3}{4}}\right]^{-2}. \]  

(16)

The ground state density moments are given by the simple expression:
\[ <r^k>_{1s} = \frac{\Gamma(k+4)}{6}, \]  

(17)

where \( \Gamma(x) \) designates Euler’s Gamma function [23]. See Table 1 for the first few values. The exact Laplace transform is found to be equal to:
\[ L(q) = \frac{1}{(1+q)^4}. \]  

(18)

The Padé approximants produce two continuous fractions: \( P(0, 3) \) and \( P(0, 4) \). \( P(0, 3) \) gives a rather good approximation while \( P(0, 4) \) is equal to the exact one (see (17)). Figure 1 shows \( r^2\rho_{1s}(r) \) calculated with the fractions \( P(0,3) \) and \( P(0,4) \), and compared to exact values.
The use of Laplace transform in the inverse problem from bound states

Fig. 1 – $r^2 \rho_{1s}(r)$ in the case of Kratzer potential. Black line: exact, red line: $P(0,3)$, and blue squares: $P(0,4)$.

Fig. 2 – Kratzer potential calculated using Fourier transform. The Padé approximant is compared to the exact values of $F(q)$. Red line: exact; black line: $P(6,10)$.

As in the case of the Coulomb potential, the Padé approximants with $D > 4$ are
equal to $P(0,4)$. Exact Laplace transform leads naturally to exact inverse Laplace transform, exact ground state wave function and potential. This result is very important by comparison with the one obtained via the Fourier transform. The latter is displayed in Fig. 2, together with the exact potential. The difference is noticeable at short distances. Moreover the Fourier transform requires the knowledge of eight moments [24]. Here, the Laplace transform requires the knowledge of two excited states to get the exact potential.

3.3. HULTHÉN POTENTIAL

The Hulthén potential has the expression [22]:

$$V(r) = -V_0 \frac{e^{-\lambda r}}{(1-e^{-\lambda r})}, \quad (19)$$

In this work we have considered $V_0 = \lambda = 0.5$. The ground state reduced wave function is given by:

$$\chi_{1s}(r) = N_{1s} e^{-\alpha r} (1 - e^{-\lambda r}). \quad (20)$$

The moments are calculated using:

$$<r^k>_{1s} = N(k)/N_{1s}, \quad (21)$$

where

$$N(k) = \Gamma(k+1) \left[ \frac{1}{(2\alpha)^{k+1}} - \frac{2}{(2\alpha + \lambda)^{k+1}} + \frac{1}{(2\alpha + 2\lambda)^{k+1}} \right], \quad (22)$$

and $N_{1s}$ is the normalization constant and $\alpha$ is a parameter linked to the ground state energy by $\alpha = \sqrt{-2E_{1s}/\lambda^2}$. The first moments are given in Table 1. The energy spectrum takes the form:

$$E_{n,\ell=0} = -V_0 \left[ \frac{\beta^2 - n^2}{2n\beta} \right] ; \beta^2 = \frac{2V_0}{\lambda^2} ; n = 1, 2, ... \quad (23)$$

The exact Laplace transform of the ground state density is given by:

$$L(q) = \frac{30}{30 + 47q + 24q^2 + 4q^3}. \quad (24)$$

As for the Coulomb potential, we find the fraction $P(0,3)$ to yield the exact Laplace transform (24). It leads naturally to the exact potential. When use is made of the Fourier transform of the ground state density, the exact result is given by the fraction $P(0,4)$. 
Fig. 3 – Harmonic oscillator. The exact Laplace transform (black full line) is compared to the first Padé approximants. red: $P(0, 3)$; green: $P(0, 4)$; blue: $P(0, 5)$; cyan: $P(0, 6)$; yellow: $P(1, 4)$; magenta: $P(0, 7)$; orange: $P(0, 8)$.

3.4. HARMONIC OSCILLATOR

If we consider $\omega = 1$, the harmonic oscillator potential is defined by:

$$V(r) = \frac{1}{2} r^2$$

The ground state radial wave function takes the form:

$$R_{1s}(r) = \frac{2}{\pi^{1/4}} e^{-r^2/2}.$$

The exact ground state density moments are expressed as:

$$<r^k>_{1s} = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{k + 3}{2}\right).$$

The energies are given by:

$$E_{n,\ell} = (2n + \ell + \frac{3}{2}); n = 0, 1, 2, \ldots; \ell = 0, 1, 2, \ldots$$

The exact Laplace transform takes the form:

$$L(q) = \frac{1}{2\sqrt{\pi}} \left( \sqrt{\pi} e^{-q^2/4} (q^2 + 2) [1 - erf\left(\frac{q}{2}\right)] - 2q \right).$$
Here, \( erf(x) \) designates the error function [23]. The lowest Padé approximants are displayed in Fig. 3, and compared to exact Laplace transform given in (28). Fractions containing singularities have been eliminated, because they forbid the integration determining the inverse Laplace transform. All fractions approximate the exact expression to a satisfactory level, even \( P(0,3) \). Actually, for small \( q \), all fractions are similar. Differences appear at large \( q \) (see the inserted enlargement in Fig. 3). The accuracy of the fit grows with the order of the Padé approximant. Averaging the chosen Padé approximants provides us with an excellent fit to the exact Laplace transform, as shown in Fig. 4.

The square of the reduced ground state wave function \( r^2 \rho(r) \) is displayed in Fig. 5 for few Padé approximants. Note that use has been made of Maple software and numerical integration based on the Fourier series expansion. In some cases, the Fortran program we are using, do not allow us to calculate the reduced wave function values near the origin (for instance \( 0 < r < 0.4 \) in the case of \( P(0,5) \)). This difficulty can be avoided: different ways to estimate the value of \( \rho(r) \) at the origin have been discussed [12, 13]. Interpolation can then be used to reach the missing part.

The radial wave function is displayed in Fig. 6 for the set of chosen approximants. It shows the dispersion of the curves near the origin. This encountered uncertainty is linked to the uncertainty of the Laplace transform at large \( q \), as underlined in previous works using the Fourier transform [15, 24]. It can be cured by averaging
a great number of fractions.

The potential derived from different Padé approximants are displayed in Fig. 7. At first glance, the results are not very satisfactory. Near the origin \( r = 0 \), the potential is undetermined, and diverges. As stated before, this problem is directly linked to the Laplace transform at large distances, which requires the knowledge of a great number of ground state density moments [15, 24]. Nevertheless, a good approximation of \( \rho(0) \) can be reached by a method based on the Stieltjes moment problem [13]. This method needs the moments of order \( k = -1 \) and \( k = -2 \) of the ground state density. They can be evaluated via the Laplace transform. We recall that:

\[
<r^{-1}> = \int_0^\infty L(q)dq.
\] (29)

\[
<r^{-2}> = \int_0^\infty L(q)qdq.
\] (30)

![Graph](image)

Fig. 5 – Squared reduced wave function of the harmonic oscillator \( r^2\rho_{1s}(r) \). Black squares: exact; green line: \( P(0,3) \); blue line: \( P(0,4) \); cyan triangles: \( P(0,5) \); red line: \( P(1,4) \).

In Table 2, we present the evaluation of these moments by using \( P(0,3) \), \( P(0,4) \), and their average. The results are satisfactory in the case of the harmonic oscillator. They are improved by considering the average of a great number of fractions. By doing so we reached the rather precise value of 1.1284 for \( <r^{-1}>_{1s} \), while \( <r^{-2}>_{1s} \) is found within a precision of 2\%.
Table 2

$<r^{-1}>_{1s}$ and $<r^{-2}>_{1s}$ calculated using Padé approximant fractions $P(0.3)$, $P(0.4)$ and their average, in the case of studied potentials.

<table>
<thead>
<tr>
<th>Potential</th>
<th>$&lt;r^{-1}&gt;$</th>
<th>$&lt;r^{-2}&gt;$</th>
<th>$\frac{1}{\sinh^2(r)}$</th>
<th>$\frac{10}{\cosh^2(r)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;r^{-1}&gt;$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>1.1283</td>
<td>0.9681</td>
<td>1</td>
<td>0.3333</td>
</tr>
<tr>
<td>P(0.3)</td>
<td>1.1671</td>
<td>0.9681</td>
<td>0.9999</td>
<td>-</td>
</tr>
<tr>
<td>P(0.4)</td>
<td>1.1139</td>
<td>-</td>
<td>-</td>
<td>0.3333</td>
</tr>
<tr>
<td>Average</td>
<td>1.1405</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$&lt;r^{-2}&gt;$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>2.00</td>
<td>1.8950</td>
<td>2</td>
<td>1.1666</td>
</tr>
<tr>
<td>P(0.3)</td>
<td>2.3221</td>
<td>1.8950</td>
<td>1.9999</td>
<td>-</td>
</tr>
<tr>
<td>P(0.4)</td>
<td>1.7532</td>
<td>-</td>
<td>-</td>
<td>0.1666</td>
</tr>
<tr>
<td>Average</td>
<td>2.0376</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 6 – Harmonic oscillator ground state radial wave function $R_{1,0}(r)$. Black squares: exact; green line: $P(0,3)$; blue line: $P(0,4)$; red line: $P(1,4)$.

On the other side, the oscillation of the potential observed for large $r$ is due to numerical integration. Difficulties arise when inverting the Laplace transform numerically. This is a delicate step. It can be overcome by considering different algorithms. This point deserves further investigations.

The harmonic oscillator shows this peculiarity of a Laplace transform qualita-
tively well reproduced by Padé approximants of low order. On the other hand, the potential is recovered only with the average over a large number of higher order approximants. This situation sounds in contrast to the linearity of the spectrum both in $\ell$ and $n$. Actually it tells us that this linearity cannot be asserted with confidence over the whole spectrum from a small number of states.

![Fig. 7 – Harmonic oscillator potential given by different Padé approximants. Empty squares: exact; red line: $P(0,3)$; green line: $P(0,4)$; blue line: $P(1,4)$.

3.5. A NUMERICAL EXAMPLE

Analytical and semi-analytical examples are useful to fix the method. Here, we shall treat a purely numerical case, closer in a sense to experimental situations. For this purpose, we consider the potential:

$$V(r) = \frac{1}{\sinh^2(r)} - \frac{10}{\cosh^2(r)}. \quad (31)$$

This potential has similarities with the Kratzer potential at short distances, and simulates the Hulthén potential at large $r$. Note that the Schrödinger equation has to be solved up to large $r$, in order to reach sufficiently precise values of the ground state density.

The first moments of the ground state density are given in Table 1. The reduced wave function calculated by using different fractions is shown in Fig. 8, and compared to the “exact” numerical one. As we can see, the fraction $P(0,3)$ must be
Fig. 8 – The Laplace transform of the potential $\frac{1}{\sinh^2(r)} - \frac{10}{\cosh^2(r)}$. Exact: black squares; magenta line: $P(0, 3)$; green line: $P(0, 4)$; blue line $P(0, 5)$; cyan squares: $P(0, 6)$, and yellow line: $P(0, 7)$.

Fig. 9 – The potential $\frac{1}{\sinh^2(r)} - \frac{10}{\cosh^2(r)}$ calculated with Padé approximants $P(0, 4)$ (green), $P(0, 5)$ (blue), $P(0, 6)$ (red), and $P(0, 7)$ (yellow), compared to the exact one (black squares).

rejected because the inversion of Laplace transform produces negative values of the ground state density. The other Padé approximants gives a reduced wave function
close to the exact one. The fraction \( P(0, 7) \) does not give a complete wave function near the origin, because the Fortran code we have used do not allows us to estimate the inverse Laplace transform at short distances.

In Fig. 9, we display the potential produced by different fractions, and compare them to the exact potential. We note the uncertainties near the origin, a problem previously discussed, notably in the case of the Kratzer potential [15, 24]. Nevertheless, the fractions \( P(0, 5) \) and \( P(0, 6) \) give a good behavior even near \( r = 0 \). As stated before, the potential is only determined for a limited distance range. This is directly related to the difficulty of numerically calculating the inverse Laplace transform for the considered fractions.

4. CONCLUSIONS

In this paper, we discuss the inverse problem in the case of bound states. We consider the Schrödinger equation in the \( D = 3 \) dimensional space with spherical symmetry for a local potential. Our method uses a recurrent relationships between the moments of the ground state density \( \langle r^k \rangle_{1s} \) and the energies of the lowest state of each angular momentum \( E_{1\ell} \). In previous works, the reconstruction of the ground state wave function was based on the Fourier transform \( F(q) \) of the ground state density. A good accuracy is reached when the number of moments is reasonably high (15-20). It constitutes a serious limitation in the case of experimental data where it is rarely possible to have sufficient energies and therefore moments. Here, we have investigated the possibility of substituting \( F(q) \) by the Laplace transform \( L(q) \) of \( r^2 \rho(r) \). Our aim is to verify if this substitution leads to equivalent results but with less number of moments. Here, we have investigated in the three cases of Coulombic, Hulthén, and Kratzer potentials. We have used the fraction \( P(0, 3) \) in the first and the second cases and \( P(0, 4) \) in the third one, respectively. We recall that in the case of Fourier transform, a good aproximation requires the \( P(6, 10) \) approximant for the Kratzer potential.

In the harmonic oscillator case, the Laplace transform bring new information but whatever the procedure, a large amount of moments is needed to recover the potential. This situation is related to the strict linearity of the spectrum, which requires a larger number of known states to be asserted. On the other hand, because the Laplace transform gives accesss to the \( k = -1 \) and \( k = -2 \) ground state density moments, it offers a way to determine the value of the density at the origin via the method known as the Stieltjes moment problem [13].

The numerical example shows that the fractions \( P(0, 4) \) and \( P(0, 5) \) are closed to the exact (numerical) Laplace transform and give a good approximation to the exact potential.
Finally, the use of the Laplace transform brings valuable information to reconstruct the potential from data. It takes advantage of smaller number of ground state density moments necessary to reach a sufficient accuracy. Combined with the Fourier transform approach, it constitutes an efficient method for the inverse problem in the case of bound state. In future, this method can be used to reconstruct the effective potential in the case of hadronic atoms and then to deduce its nuclear part.

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