A (3+1)-DIMENSIONAL INTEGRABLE CALOGERO-BOGOYAVLSENSKII-SCHIFF EQUATION AND ITS INVERSE OPERATOR: LUMP SOLUTIONS AND MULTIPLE SOLITON SOLUTIONS

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Abstract. In this work, we built a (3+1)-dimensional integrable equation. We started by reformulating the main equation of our model by combining the recursion operator of the Calogero-Bogoyavlenskii-Schiff equation with its inverse recursion operator. We confirm the complete integrability of our new developed equation by demonstrating that it satisfies the Painlevé property. We get a variety of lump solutions that are obtained under specific constraints. Furthermore, we used the simplified Hirota’s direct approach to find multiple soliton solutions to the new evolution equation. In addition, other techniques are used to solve the new evolution equation, in order to get some physically relevant solutions.

Key words: Calogero-Bogoyavlenskii-Schiff equation, lump solutions, multiple soliton solutions, Painlevé analysis.

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1. INTRODUCTION

The studies of nonlinear partial differential equations (PDEs) are intensively growing among scientific, engineering, and technical communities [1–33], because these equations furnish the substantial attributes of the scientific developments. The dispersion, dissipation, diffusion, reaction and convection phenomena play a significant role in nonlinear wave applications. There is no unified technique to handle nonlinear differential equations due to its structure complexity. A variety of analytical and numerical methods has been employed for analyzing different types of evolution equations. The integrability of nonlinear PDEs has received a lot of attention. The Painlevé test is an effective method for checking the integrability and the Painlevé property of these PDEs.
The recursion operator is an integro-differential operator that connects two consecutive symmetries [34–44]. The recursion operator of a nonlinear equation indicates that this equation has infinitely many higher-order symmetries, which is a strong feature of its complete integrability as discovered by Olver [6].

In what follows, we give a brief summary of our work in [1], and we plan to extend that work. The hereditary symmetry \( \Phi(u(x,t)) \) is a recursion operator of the following hierarchy of evolution equations

\[
\frac{\partial_{t}}{\partial_{x}} + \Phi(u)u_{x} = 0,
\]

which generates a range of (1+1)-dimensional equations. The recursion operator \( \Phi(u) \) to the Korteweg-de Vries (KdV) equation reads

\[
\Phi(u) = \partial_{x}^{2} + 4u + 2u_{x}\partial_{x}^{-1},
\]

where \( \partial_{x} \) and \( \partial_{x}^{-1} \) indicate, respectively, the total derivative and its integral operator with respect to \( x \).

However, the following integrable Calogero-Bogoyavlenskii-Schiff (CBS) equation

\[
\frac{\partial_{t}}{\partial_{x}} + u_{xxy} + 4uu_{y} + 2u_{x}\partial_{x}^{-1}(u_{y}) = 0,
\]

is determined via the hierarchy of the following equation

\[
\frac{\partial_{t}}{\partial_{x}} + \Phi(u)u_{y} = 0,
\]

where \( \Phi(u) \) has the same form (2) as that for the KdV equation with argument \( x \).

Olver’s work [6] was investigated deeply by Verosky [9], who acknowledged the use of the negative direction to produce a series of equations with progressively negative orders. According to the investigation made by Verosky [9], the hierarchy of evolution Eq. (1)

\[
\frac{\partial_{t}}{\partial_{x}} = -\Phi(u_{x}),
\]

can be used in the negative order hierarchy in the form

\[
\frac{\partial_{t}}{\partial_{x}} = -\Phi^{-1}u_{x},
\]

where the negative power of \( \Phi \) just represents the opposite direction [7–9]. In other words, the negative order equation is expressed as

\[
\Phi(\frac{\partial_{t}}{\partial_{x}}) = -u_{x}.
\]

This research has two objectives. First, we aim to create a novel (3+1)-dimensional integrable equation, by merging the senses of the CBS recursion operator (2) with the negative-order recursion operator (7). To verify the complete integrability of the new developed equation, the Painlevé analysis is applied for this propose. Secondly, we aim to determine multiple soliton solutions from the new integrable equation that we made.
2. FORMULATION OF THE (3+1)-INTEGRABLE COMBINED EQUATION

To create a novel (3+1)-dimensional integrable equation, we will mix the CBS recursion operator (1) with the negative-order recursion operator (2), which leads to

\[ v_t + \Phi(v)(v_x + v_y + v_z) + \Phi(v)v_t = 0, \]  

or equivalently

\[ v_t + \Phi(v)(v_x + v_y + v_z + v_t) = 0, \]  

where \( v \equiv v(x, y, z, t) \). The combined CBS equation with its negative-order form can be obtained by applying the recursion operator as specified in (2), which give us the following (3+1)-dimensional combined CBS-negative-order CBS (CBS-nCBS) equation

\[ v_t + (v_x + v_y + v_z + v_t)x + 4v(v_x + v_y + v_z + v_t) + 2v_x\partial_x^{-1}(v_x + v_y + v_z + v_t) = 0. \]  

To get ride of the integral operator, we equate the potential to

\[ v = u_x, \]  

and will carry out (10) to

\[ u_{xt} + (u_x + u_y + u_z + u_t)xx + 4u_x(u_x + u_y + u_z + u_t)x + 2u_{xx}(u_x + u_y + u_z + u_t) = 0, \]  

where \( u \equiv u(x, y, z, t) \). Moreover, we will extend (12) to

\[ u_{xt} + (u_x + u_y + u_z + u_t)xxx + 4u_x(u_x + u_y + u_z + u_t)x + 2u_{xx}(u_x + u_y + u_z + u_t) + (\alpha u_x + \beta u_y + \gamma u_z)_x = 0, \]  

where three linear terms, namely \( \alpha u_{xx}, \beta u_{xy}, \) and \( \gamma u_{xz} \) were added to (12), and \( \alpha, \beta, \gamma \neq 1 \).

Note that Eq. (13) has eight linear dispersive terms as well as eight nonlinear terms. In the following Section, the complete integrability of Eq. (13) will be explored by exemplifying that it passes the Painlevé test neatly.

3. THE PAINLEVÉ TEST

The Painlevé test will be applied to Eq. (13) using the Weiss-Tabor-Carnevale (WTC) method with Kruskal's simplification to examine its Painlevé integrability [5]. The WTC-Kruskal method [5] is summarized in the following three steps:

(i) leading-order analysis,
(ii) finding resonances, and
(iii) identifying compatibility conditions. More details about the three steps are given in Refs. [1–10].
In order to check the integrability of Eq. (13), the solution of Eq. (13) is assumed to be given by the following Laurent series

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\mu}(x, y, z, t),$$

(14)

with a sufficient number of arbitrary functions $u_j \equiv u_j(x, y, z, t)$ in addition to $\phi \equiv \phi(x, y, z, t)$. The Painlevé property admits resonances when $\mu$ takes negative integer values and $j$ takes positive integers ones. First, we need to find both leading order and coefficient $\mu$ and $u \equiv u_0(x, y, t)$, respectively. To do that, we insert

$$u = u_0 \phi^\mu,$$

(15)

into Eq. (13). By balancing both dispersive and nonlinear terms, we have

$$\mu = -1, \quad u_0 = 2 \phi_x.$$

(16)

The resonance at $j = -1$ indicates to the arbitrariness of the singular manifold $\phi(x, y, z, t) = 0$.

In the next step, we check the presence of a sufficient number of arbitrary functions as well as to obtain the resonance points. To accomplish this, we introduce the Laurent series

$$u = u_0 \phi^{-1} + u_j \phi^{j-1}, \quad j \geq 1,$$

(17)

into Eq. (13), along with (16), to determine the characteristic equation for resonances

$$(j + 1)(j - 1)(j - 4)(j - 6) = 0.$$ 

(18)

The characteristic equation has solutions (resonances $j$) at $j = -1, 1, 4,$ and $6$. Recall that the resonance at $j = -1$ corresponds to the arbitrariness of the singular manifold $\phi(x, y, z, t) = 0$. Notice that the resonance points are not affected by the additional terms $\alpha u_{xx}, \beta u_{xy},$ and $\gamma u_{xz}$. The role of these additional terms will be clear in the dispersion relation and on the constraints of the existence of lump solutions as will be seen later. The obtained result is identical to the result obtained in our previous work [1], hence we skip details. The combined CBS-nCBS Eq. (13) allows an adequate number of arbitrary functions, which means it is integrable in the sense that it exhibits the Painlevé property.

4. MULTIPLE SOLITON SOLUTIONS

The dispersion relation (DR) of Eq. (13) can be obtained by inserting

$$u = e^{\theta_i} = e^{k_i x + r_i y + s_i z - \omega_i t},$$

(19)
into the linear terms of Eq. (13), which gives the following DR \( c_i \) as
\[
c_i = \frac{k_i^2 (k_i + r_i + s_i) + (\alpha k_i + \beta r_i + \gamma s_i)}{k_i^2 + 1}.
\] (20)

Consequently, the variable of dispersion reads
\[
\theta_i = k_i x + r_i y + s_i z - \frac{k_i^2 (k_i + r_i + s_i) + (\alpha k_i + \beta r_i + \gamma s_i)}{k_i^2 + 1} t.
\] (21)

After that the following transformation is used
\[
u = 2 (\ln f)_x,
\] (22)
where the auxiliary function (AF) \( f \equiv f(x, y, z, t) \). For one-soliton solution the AF \( f_1 \) reads
\[
f_1 = 1 + e^{\frac{k_1 x + r_1 y + s_1 z - \frac{k_2 (k_1 + r_1 + s_1) + (\alpha k_1 + \beta r_1 + \gamma s_1)}{k_1^2 + 1} t}{k_1^2 + 1}}.
\] (23)

Inserting Eqs. (22) and (23) into Eq. (13), we get one-soliton solution
\[
u = \frac{2k_1 e^{k_1 x + r_1 y + s_1 z - \frac{k_2 (k_1 + r_1 + s_1) + (\alpha k_1 + \beta r_1 + \gamma s_1)}{k_1^2 + 1} t}}{1 + e^{\frac{k_1 x + r_1 y + s_1 z - \frac{k_2 (k_1 + r_1 + s_1) + (\alpha k_1 + \beta r_1 + \gamma s_1)}{k_1^2 + 1} t}}},
\] (24)
Remember that the potential specified earlier in Eq. (11) is used to calculate \( v \equiv v(x, y, z, t) \)

For the two soliton solutions, we adjust the AF \( f_2 \) as
\[
f_2 = f_1 + e^{\theta_1 + a_{12} e^{\theta_1 + \theta_2}},
\] (25)
where \( \theta_i, i = 1, 2 \) are defined in Eq. (21), and \( a_{12} \) indicates the phase shift, which will be determined later. Using Eq. (25) into Eq. (13), and solving the obtained equation, we get
\[
a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},
\] (26)
which is generalized as
\[
a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \leq i < j \leq 3.
\] (27)

This is identical to the phase shift of the standard CBS equation [1–6]. It should be emphasized that the phase shifts \( a_{ij} \) do not rely on \( r_i, s_i, \alpha, \beta, \) and \( \gamma \), but remain constant for both the regular CBS and the combined CBS-nCBS Eq. (13).

For the three soliton solutions, the AF \( f_3 \) reads
\[
f_3 = f_2 + e^{\theta_1 + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + a_{12} a_{23} a_{13} e^{\theta_1 + \theta_2 + \theta_3}},
\] (28)
where the values of \( \theta_i, i = 1 - 3 \) are defined in Eq. (21). Proceeding as before gives three soliton solutions for the combined CBS-nCBS equations (13).
5. Variety of Lump Solutions

To generate various lump solutions for Eq. (13), initially, we convert it into the corresponding Hirota bilinear operators form

\[ (D_xD_t + D_x^4 + D_x^3D_y + D_x^3D_z + D_x^2D_t + \alpha D_x^2 + \beta D_xD_y + \gamma D_xD_z) f \cdot f = 0. \]  
(29)

Here, \( D_t, D_x, D_y, \) and \( D_z \) represent the Hirota’s bilinear derivative operators. Accordingly, Eq. (13) will be converted to

\[
(f_{xt} - f_t f_x) + (f_{xxxx} - 4f_x f_{xxx} + 3f_{xx} f_{xx}) \\
+ (f_{xxy} - 3f_{xxy} + 3f_{xx} f_{xy} - f_{xxx} f_y) \\
+ (f_{xxxx} - 3f_{xxy} + 3f_{xx} f_{xy} - f_{xxx} f_x) \\
+ \alpha(f_{xxx} - f_x f_x) + \beta(f_{xy} - f_x f_y) + \gamma(f_{xx} - f_x f_x) = 0,
\]
(30)

obtained upon using

\[ u = 2(\ln f)_x. \]
(31)

Now, for obtaining the lump solutions to Eq. (13), the following hypotheses are considered

\[
g = a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\
h = a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\
f = g^2 + h^2 + a_{11},
\]
(32)

where \( a_j, 1 \leq j \leq 11 \) are undetermined real parameters. To find the values of \( a_j, 1 \leq j \leq 11 \), we insert Eq. (32) into Eq. (30), and after several long but simple calculations, we get a system of algebraic equations, and by solving this system with the help of Maple, we obtain the set of constraint equations:

Case 1.

To study the first case, we set

\[
a_1 = a_i, i = 1, 4, 5, 6, 7, 9, 10, 11, \\
a_2 = \frac{(\alpha - \gamma)(a_1^2 + a_8^2) + a_6a_9(\beta - 2\gamma + 1) - a_1a_4(\gamma - 1)}{(3\gamma - \gamma)(a_1)}, \beta \neq \gamma, a_1 \neq 0, \\
a_3 = \frac{\alpha(\gamma a_1^2 + \beta a_8^2) + \beta a_6a_9(\beta - \gamma) - \beta(\beta^2 + a_8^2) - \gamma a_4(\beta - 1) - \beta a_6a_9(\gamma - 1)}{\gamma(\beta - \gamma)a_1}, \\
\beta \neq \gamma, a_1 \neq 0, \gamma \neq 0, \\
a_8 = -\frac{\alpha a_6 + \beta a_7 + a_9}{\gamma},
\]
(33)

where \( a_{11} > 0 \), to ensure the positivity for the function \( f(x, y, z, t) \) and the localization of \( u(x, y, z, t) \) in all directions of the space, respectively. The parameters values given in Eq. (33) create the class of positive quadratic function solutions on substitution Eq. (33) into Eq. (32). However, a class of lump solution can be obtained under the following condition

\[ \beta = -2\alpha + 4\gamma - 1. \]
(34)
This will give a class of lump solutions to Eq. (13) by using \( u = 2(\ln f(x, y, z, t))_x \) in Eq. (32). Moreover, the obtained lump solutions \( u(x, y, z, t) \to 0 \) exist if and only if \( g^2 + h^2 \to \infty \).

For example, selecting

\[
\begin{align*}
    a_1 &= 2, a_4 = 2, a_5 = 1, a_6 = 2, a_7 = 1, a_9 = 1, a_{10} = 2, a_{11} = 3, \\
    \alpha &= 2, \beta = 7, \gamma = 3
\end{align*}
\]

(35)
gives

\[
\begin{align*}
    a_2 &= \frac{3}{2}, a_3 = -\frac{11}{2}, a_8 = -4.
\end{align*}
\]

(36)

and the other parameters are the same as before, then we get the following lump solution

\[
\begin{align*}
    u &= \frac{2(16x + 10y - 38z + 12t + 12)}{(2x + \frac{3}{2}y - \frac{11}{2}z + 2t + 1)^2 + (2x + y - 4z + t + 2)^2 + 3}.
\end{align*}
\]

(37)

**Case 2.** In this case, we set

\[
\begin{align*}
    a_i &= a_i, i = 1, 2, 3, 5, 6, 9, 10, 11, \\
    a_4 &= -(\alpha a_1 + \beta a_2 + \gamma a_3), \\
    a_7 &= \frac{\alpha(\gamma a_1^2 + a_2^2) + \gamma a_1 a_2(\beta - 1) + \gamma a_1 a_3(\gamma - 1) + a_2^2(\alpha - \gamma) - a_6 a_9(\gamma - 1)}{(\beta - \gamma)a_6}, \beta \neq \gamma, a_6 \neq 0, \\
    a_8 &= \frac{\alpha(\beta a_1^2 + a_2^2) + \beta a_1 a_3(\gamma - 1) + \beta a_1 a_2(\beta - 1) + a_2^2(\alpha - \beta) - a_6 a_9(\beta - 1)}{(\beta - \gamma)a_6}, \beta \neq \gamma, a_6 \neq 0,
\end{align*}
\]

(38)

where \( a_{11} > 0 \), for the same reason mentioned above. The obtained parameters (38) create the class of positive quadratic function solutions on substitution of Eq. (38) into Eq. (32). However, a class of lump solution can be obtained under the following constraint

\[
\gamma = 5 - 2\alpha - 2\beta.
\]

(39)

This gives a class of lump solutions to Eq. (13) by using \( u = 2(\ln f(x, y, z, t))_x \) in Eq. (32). Moreover, the obtained lump solutions \( u(x, y, z, t) \to 0 \) exist if and only if \( g^2 + h^2 \to \infty \).

A numerical example: by choosing

\[
\begin{align*}
    a_1 &= 2, a_2 = 2, a_3 = 1, a_5 = 1, a_6 = 2, a_7 = 1, a_9 = 1, a_{10} = 2, a_{11} = 3, \\
    \alpha &= 1, \beta = 2, \gamma = -1
\end{align*}
\]

(40)
gives

\[
\begin{align*}
    a_4 &= -5, a_7 = -2, a_8 = -1.
\end{align*}
\]

(41)

and the other parameters are the same as before, then we get the following lump
solution

\[ u = \frac{2(16x - 16t + 12)}{(2x + 2y + z - 5t + 1)^2 + (2x - 2y - z + t + 2)^2 + 3}. \]  

(42)

Case 3.

We close the lump solution by studying the following case

\[ a_i = a_i, i = 1, 4, 5, 6, 7, 8, 10, 11, \]
\[ a_2 = -\frac{(\alpha - \gamma)(a_7^2 + a_8^2) + a_6a_7(\beta - 1) + \gamma a_6a_8(\gamma - 1) - a_4a_7(\gamma - 1)}{(\beta - \gamma)a_1}, \beta \neq \gamma, a_1 \neq 0, \]
\[ a_3 = \frac{a_6a_7 + \beta a_7(\beta - 1) + a_5(\alpha - \gamma) - a_4a_7(\beta - 1)}{(\beta - \gamma)a_1}, \beta \neq \gamma, a_1 \neq 0, \gamma \neq 0, \]
\[ a_9 = -(\alpha a_5 + \beta a_7 + \gamma a_8), \]

(43)

where \( a_{11} > 0 \), for the same reason mentioned above. The obtained results given in Eq. (43) generate the class of positive quadratic function solutions by inserting Eq. (43) into Eq. (32). However, a class of lump solution can be obtained under the following condition

\[ \beta = 5 - 2\alpha - 2\gamma. \]  

(44)

This gives a class of lump solutions to Eq. (13) by using \( u = 2(\ln f(x, y, z, t))_x \) in (32). Moreover, the obtained lump solutions \( u(x, y, z, t) \to 0 \) exist if and only if \( g^2 + h^2 \to \infty \).

For example, selecting
\[ a_1 = 2, a_4 = 2, a_5 = 1, a_6 = 2, a_7 = 1, a_9 = 1, a_{10} = 2, a_{11} = 3, \alpha = 2, \beta = 7, \gamma = 3 \]

(45)

gives
\[ a_2 = -\frac{4}{3}, a_3 = -\frac{b}{3}, a_9 = -5, \]

(46)

and the other parameters are the same as before, then we get the following lump solution

\[ u = \frac{2(16x - \frac{4}{3}y - \frac{8}{3}z - 12t + 12)}{(2x - \frac{4}{3}y - \frac{8}{3}z + 2t + 1)^2 + (2x + y + 2z - 5t + 2)^2 + 3}. \]  

(47)

In the following parts, we will use a collection of distinct ansatze to find other precise solutions with distinct physical properties. The methods that we will employ have been described in depth in the literature [28–44].
6. OTHER SOLUTIONS

In this part, we will apply some well-known methods to find alternative solutions to our new developed Eq. (13)

6.1. THE EXPONENTIAL METHOD

Based on the rational tanh method, the following ansatz is introduced

\[ u = a_0 + a_1 e^{kx+ry+sz-Rt}. \]  

(48)

Substituting this assumption into Eq. (13), and proceeding as in [1–4], we find

\[ R = \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}, \beta \neq 1, \]
\[ r = \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}. \]  

(49)

This then yields the traveling wave solution

\[ u = a_0 + a_1 e^{kx+\left(\frac{1-\alpha)k+(1-\gamma)s}{\beta-1}\right) y+sz-\frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1} t}. \]  

(50)

6.2. RATIONAL OF EXPONENTIAL METHOD I

We can assume that Eq. (13) supports the following solution

\[ u = \frac{1}{a_0 + a_1 e^{kx+ry+sy-Rt}}. \]  

(51)

Substituting the assumption (51) into Eq. (13), and after several simple manipulations, we obtain the first set of parameters

\[ R = \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}, \beta \neq 1, \]
\[ r = \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}. \]  

(52)

and the second set of parameters is found to be

\[ a_0 = -\frac{1}{2k}, k \neq 0, \]
\[ R = \frac{4k^2(k+r+s)+(\alpha k+\beta r+\gamma s)}{k^2+1}, \]  

(53)

where the remaining parameters are arbitrary. Accordingly, the following solutions are obtained

\[ u = \frac{1}{a_0 + a_1 e^{\left(kx+\frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}\right) y+sz-\frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1} t}}, \]  

(54)

and

\[ u = \frac{1}{-\frac{1}{2k} + a_1 e^{\left(kx+ry+sz-\frac{4k^2(k+r+s)+(\alpha k+\beta r+\gamma s)}{4k^2+1} t}}. \]  

(55)
6.3. RATIONAL OF EXPONENTIAL METHOD II

Next, we apply the following solution

$$u = \frac{b_0 + b_1 e^{kx + ry + sy - Rt}}{a_0 + a_1 e^{kx + ry + sy - Rt}}. \tag{56}$$

Collecting the exponential functions coefficients and substituting the assumption (56) into Eq. (13), we get

$$b_0 = -\frac{a_0 (2a_1 k - b_1)}{a_1},$$

$$R = \frac{4k^2 (k + r + s) + (\alpha k + \beta r + \gamma s)}{k^2 + 1}, \tag{57}$$

where the remaining parameters are left free. Substituting the result (57) into the solution (56), the traveling wave solution is obtained.

6.4. SECH/SEC METHOD

The sech technique permits using the solution as

$$u = a_0 + a_1 \text{sech} (kx + ry + sy - Rt). \tag{58}$$

Inserting the assumption (58) into Eq. (13) and after several simplified calculations the following result is obtained

$$R = \frac{(\beta - \alpha)k + (\beta - \gamma)s}{\beta - 1}, \beta \neq 1,$$

$$r = \frac{(1 - \alpha)k + (1 - \gamma)s}{\beta - 1}, \tag{59}$$

where the other parameters remain free. This provides the following soliton solution

$$u = a_0 + a_1 \text{sech} (kx + \frac{(1 - \alpha)k + (1 - \gamma)s}{\beta - 1} y + sz - \frac{(\beta - \alpha)k + (\beta - \gamma)s}{\beta - 1} t). \tag{60}$$

In a like manner, we can show that

$$u = a_0 + a_1 \text{sec} (kx + \frac{(1 - \alpha)k + (1 - \gamma)s}{\beta - 1} y + sz - \frac{(\beta - \alpha)k + (\beta - \gamma)s}{\beta - 1} t). \tag{61}$$

6.5. TANH/COTH METHOD

According to the tanh technique, the following solution is introduced

$$u = a_0 + a_1 \tanh (kx + ry + sy - Rt). \tag{62}$$

Now, by substituting the ansatz (62) into Eq. (13), we finally get

$$R = \frac{(\beta - \alpha)k + (\beta - \gamma)s}{\beta - 1}, \beta \neq 1,$$

$$r = \frac{(1 - \alpha)k + (1 - \gamma)s}{\beta - 1}, \tag{63}$$
and the second set of parameters is found to be

\[
\begin{align*}
a_1 &= 2k, \\
R &= \frac{4k^2(k+r+s)+(\alpha k+\beta r+\gamma s)}{4k^2+1},
\end{align*}
\]

Consequently, we get the following two soliton solutions

\[
u = a_0 + a_1 \tanh (kx + \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}y + sz - \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}t), \quad (65)
\]

and

\[
u = a_0 + 2k \tanh (kx + ry + sz - \frac{4k^2(k+r+s)+(\alpha k+\beta r+\gamma s)}{4k^2+1}t). \quad (66)
\]

In a like manner, we can show that

\[
u = a_0 + a_1 \coth (kx + \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}y + sz - \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}t), \quad (67)
\]

and

\[
u = a_0 + 2k \coth (kx + ry + sz - \frac{4k^2(k+r+s)+(\alpha k+\beta r+\gamma s)}{4k^2+1}t), \quad (68)
\]

are two singular solutions of the same equation.

### 6.6. RATIONAL TANH SCHEME

The following solution is established using the rational tanh scheme

\[
u = \frac{1}{a_0 + a_1 \tanh (kx + ry + sz - Rt)}. \quad (69)
\]

Now, by inserting assumption (69) into Eq. (13), we finally get

\[
\begin{align*}
R &= \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}, \beta \neq 1, \\
r &= \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1},
\end{align*}
\]

This provides the following traveling wave solution

\[
u = \frac{1}{a_0 + a_1 \tanh (kx + \frac{(1-\alpha)k+(1-\gamma)s}{\beta-1}y + sz - \frac{(\beta-\alpha)k+(\beta-\gamma)s}{\beta-1}t)}.
\]

In a like manner, we can employ distinct schemes to get the following solutions

\[
u = a_0 + a_1 \sinh (\Theta), \quad (72)
\]

\[
u = a_0 + a_1 \cosh (\Theta), \quad (73)
\]
\[ u = a_0 + a_1 \sin(\Theta), \quad (74) \]
\[ u = a_0 + a_1 \cos(\Theta), \quad (75) \]
\[ u = a_0 + a_1 \tan(\Theta), \quad (76) \]
\[ u = a_0 + a_1 \cot(\Theta), \quad (77) \]
\[ u = a_0 + a_1 \sec(\Theta), \quad (78) \]
\[ u = a_0 + a_1 \csc(\Theta), \quad (79) \]
and
\[ u = a_0 + a_1 \ln(\Theta), \quad (80) \]
with \( \Theta = (kx + ry + sz - Rt) \), where the values of the coefficients \((r, R)\) are defined in Eq. (70).

The derived soliton solutions may play a major role in elaborating the physical aspects of diverse nonlinear phenomena.

7. DISCUSSION

Both recursion and inverse recursion operators for the CBS equation were employed for furnishing a (3+1)-dimensional combined CBS equation with its negative-order form. We showed how our new equation passes the Painlevé test to verify the complete integrability of the established model.

A simplified Hirota technique was carried out to obtain multi-soliton solutions to the proposed equation. The necessary conditions of the coefficients of the three additional terms were derived to elaborate a variety of lump solutions. In addition, we demonstrated that the proposed equation yields numerous traveling wave solutions with various physical structures. The determined solutions, solitons, lumps, periodic solutions, exponential solutions, logarithmic solutions, and others are significant in investigating nonlinear dispersive wave problems.

The outcomes demonstrate the viability and strength of the suggested approaches for many mathematical physics and engineering nonlinear evolution equations.

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REFERENCES