THE TUNNELING EFFECT THROUGH SCHWARZSCHILD BARRIER FOR SPIN 1/2 PARTICLE, ANALYTICAL AND NUMERICAL STUDY

A.V. CHICHURIN\textsuperscript{1}, E.M. OVSIIUK\textsuperscript{2}, V.M. RED’KOV\textsuperscript{3}

\textsuperscript{1}Department of Mathematical Modeling, The John Paul II Catholic University of Lublin, Konstantynow 1H, 20-708 Lublin, Poland, EU
Email: achichurin@kul.pl

\textsuperscript{2}Department of Theoretical Physics and Applied Informatics, Mozyr State Pedagogical University named after I. P. Shamyakin, Studencheskaya, 28, 2247760 Mozyr, Belarus
Email: e.ovsiyuk@mail.ru

\textsuperscript{3}Department of Fundamental Interactions and Astrophysics, B.I. Stepanov Institute of Physics, National Academy of Sciences of Belarus, Nezavisimosti Ave., 68, 220072 Minsk, Belarus
Email: v.redkov@dragon.bas-net.by

Compiled November 10, 2023

For Dirac particle, the general mathematical and numerical study of the tunneling process through the effective potential barrier generated by Schwarzschild black hole geometry is done. The main accent is given to analytical construction of the exact solutions for the problem. The study is based on the use of 8 Frobenius solutions of the relevant 2-nd order radial differential equations with the complicated structure of the singular points. We construct such solutions in explicit form and prove that the power series involved in them are converged in the whole physical region of the variable: from Schwarzschild radius to infinity. Results for tunneling effect significantly differ for two situations: one when the particle falls on the barrier from inside of the black hole and another when the particle falls from outside. Mathematical structure of the derived asymptotic relations is exact, however their further study is based on numerical summing the convergent series. In calculations, the tools of the Mathematica system are used.

\textit{Key words}: Schwarzschild black hole, Frobenius solutions, tunneling effect, convergent powers series, analytical and numerical study.

\textit{PACS}: 35Q61, 83C50, 78A25

1. INTRODUCTION

The initial idea on which the present paper is based appeared many year ago in the paper [1] by Regge and Wheeler, where stability of Schwarzschild metric [2] was studied. In [1], a linearized wave equation for spin 2 field on the background of the Schwarzschild metric was derived. It was established that a corresponding radial equation has the Schrödinger’s form with potential of the barrier type. Similar results
have been obtained for particles with other values of the spin [3]–[12].

2. BASIC EQUATIONS

For Schwarzschild metric in coordinates \( x^\alpha = (t, r, \theta, \phi) \)

\[
dS^2 = \Phi dt^2 - r^2 d\theta^2 - \frac{1}{\Phi} dr^2 - r^2 \sin^2 \theta d\phi^2, \Phi = 1 - \frac{1}{r}, r \in (1, +\infty) \tag{1}
\]

the generally covariant Dirac equation [13] takes the form (we separate a special multiplier in the wave function, \( \Psi(x) = r^{-1}\Phi^{-1/4}(x)\psi(x) \))

\[
\left[ \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i\sqrt{\Phi}\gamma^3 \partial_r + \frac{1}{r} (i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + i\sigma^{12}\cos \theta}{\sin \theta}) - M \right] \psi(x) = 0. \tag{2}
\]

Solutions with spherical symmetry may be constructed within the following substitution (instead of the commonly used formalism of spinor spherical harmonics [14] or spin-weigh harmonics [15], we apply the more simple technics based of the use of Wigner \( D \)-functions [16], [17])

\[
\psi(x)_{\epsilon jm\delta} = e^{-i\epsilon t} \begin{vmatrix} f_1(r) D_{-1/2} & f_2(r) D_{+1/2} \\ \delta f_2(r) D_{-1/2} & \delta f_1(r) D_{+1/2} \end{vmatrix}. \tag{3}
\]

This substitution corresponds to diagonalization of the total angular momentum \( \vec{J}^2, J_3 \) and the spatial reflection operator. Wigner functions are designated as follows \( D_{\sigma} = D_{-m,\sigma}(\phi, \theta, 0) \); the fixed parameters \( j = 1/2, 3/2, \ldots \) and \( m \in \{-j, \ldots, +j\} \) are omitted for brevity; the parity eigenvalues are \( \Pi = \delta(-1)^{j+1}, \delta = \pm 1 \). After needed calculation we produce the system of two equations

\[
\left( \Phi \frac{d}{dr} + \frac{\nu\sqrt{\Phi}}{r} \right) f = - (\epsilon + M\sqrt{\Phi}) g, \left( \Phi \frac{d}{dr} - \frac{\nu\sqrt{\Phi}}{r} \right) g = +(\epsilon - M\sqrt{\Phi}) f, \tag{4}
\]

where the notations are used

\[
f = (f_1 + f_2), \quad g = -i(f_1 - f_2), \quad \nu = j + 1/2, \quad \nu = 1, 2, 3, \ldots.
\]

Let us transform equations (4) to the new variable

\[
\sqrt{\Phi} = +\sqrt{1 - 1/r} = x, \quad r \to 1, x \to 0, \quad r \to +\infty, x \to +1; \tag{5}
\]

the physical region for the variable is the interval \( x \in (0, 1) \). Eqs. (4) take the form

\[
\left[ \frac{x(1-x^2)^2}{2} \frac{d}{dx} + \nu x(1-x^2) \right] f = - (\epsilon - M x) g, \left[ \frac{x(1-x^2)^2}{2} \frac{d}{dx} - \nu x(1-x^2) \right] g = +(\epsilon + M x) g. \tag{6}
\]
Whence it follows a 2-nd order equation for \( f(x) \):

\[
\left[ \frac{d^2}{dx^2} + \left( \frac{1}{x} + \frac{2}{x+1} + \frac{2}{x-1} - \frac{1}{x+c} \right) \frac{d}{dx} - \frac{2(1-3x^2)}{x(1-x^2)^2} \right.
- \frac{4\nu^2}{(1-x^2)^2} + \frac{1}{x+c} \frac{2\nu}{1-x^2} + \left( \epsilon^2 - M^2x^2 \right) \frac{4}{x^2(1-x^2)^4} \right] f = 0; \quad (7)
\]

note that \( c = \epsilon/M > 1 \). The corresponding equation for function \( g(x) \) follows from (7) by simple changes

\[ f \rightarrow g, \ \nu \rightarrow -\nu, \ c \rightarrow -c. \]

To the massless case, there corresponds the limit \( c \rightarrow \infty \), so such a singular point vanishes and we have the more simple equation

\[
\left[ \frac{d^2}{dx^2} + \left( \frac{1}{x} + \frac{2}{x+1} + \frac{2}{x-1} \right) \frac{d}{dx} - \frac{2(1-3x^2)}{x(1-x^2)^2} - \frac{4\nu^2}{(1-x^2)^2} + \frac{4\epsilon^2}{x^2(1-x^2)^4} \right] f = 0.
\]

The problem under consideration becomes more clear, when e transforming equations to other variable \( r_\ast \):

\[
\Phi \frac{d}{dr} = \frac{d}{dr_\ast}, \quad r_\ast = r + \ln(r-1), \quad r_\ast \in (-\infty, +\infty), \quad (8)
\]

to points \( r = +1, +\infty \) there correspond respectively

\[ r \rightarrow +1, \ r_\ast \rightarrow -\infty; \quad r \rightarrow +\infty, \ r_\ast \rightarrow +\infty. \quad (9)\]

In this variable, the system (4) reads

\[
\left[ \frac{d}{dr_\ast} + \nu \varphi(r_\ast) \right] f = -(\epsilon + M \sqrt{\Phi}) g, \quad \left[ \frac{d}{dr_\ast} - \nu \varphi(r_\ast) \right] g = (\epsilon - M \sqrt{\Phi}) f, \quad (10)
\]

where the function \( \varphi(r_\ast) \) is determined as

\[ \varphi(r_\ast) = \frac{\sqrt{\Phi}}{r}, \quad r_\ast \rightarrow \pm\infty, \quad \varphi(r_\ast) \rightarrow 0. \quad (11)\]

The coordinate transformation \( r \rightarrow r_\ast \) is illustrated by Fig. 1 and Fig. 2. Several typical graphs for potential \( \varphi(r_\ast) \) are given in Fig. 3; additionally we show the location of maximum values of the potentials \( \varphi(r) \) and \( \varphi(r_\ast) \):

http://www.infim.ro/rrp submitted to Romanian Reports in Physics ISSN: 1221-1451
The corresponding 2-nd order equations have the structure
\[
\begin{align*}
\left[ (\epsilon - M \sqrt{\Phi})(\frac{d}{dr_*} + \nu \varphi) \frac{1}{(\epsilon - M \sqrt{\Phi})} \right] \left( \frac{d}{dr_*} - \nu \varphi \right) g + \left( \epsilon^2 - M^2 \Phi \right) g &= 0, \\
\left[ (\epsilon + M \sqrt{\Phi}) \left( \frac{d}{dr_*} - \nu \varphi \right) \frac{1}{(\epsilon + M \sqrt{\Phi})} \right] \left( \frac{d}{dr_*} + \nu \varphi \right) f + \left( \epsilon^2 - M^2 \Phi \right) f &= 0.
\end{align*}
\]
These equations can be reduced to the form
\[
\begin{align*}
\left( \frac{d^2}{dr_*^2} + P^2(r_*) \right) f &= 0, \\
\left( \frac{d^2}{dr_*^2} + Q^2(r_*) \right) g &= 0,
\end{align*}
\] 
(12)
the quantities \(P^2(r_*)\) and \(Q^2(r_*)\) represent squared effective linear momentums. Near physical points \(r \to 1, +\infty\), equations (12) simplify
\[
\begin{align*}
r \to +1, \left( \frac{d^2}{dr_*^2} + \epsilon^2 \right) f &= 0, \left( \frac{d^2}{dr_*^2} + \epsilon^2 \right) g = 0, \ f, g \sim e^{\pm i\epsilon r_*}; \\
r \to \infty, \left( \frac{d^2}{dr_*^2} + \epsilon^2 - M^2 \right) f &= 0, \left( \frac{d^2}{dr_*^2} + \epsilon^2 - M^2 \right) g = 0, f, g \sim e^{\pm i\sqrt{\epsilon^2 - M^2} r_*}.
\end{align*}
\]
Here we have a problem where the quantum-mechanical tunneling effect is possible [1].

### 3. FROBENIUS SOLUTIONS

Let us turn to analytical consideration of possible solutions for equation (7). It is convenient to apply the following form of this equation
\[
\begin{align*}
\frac{d^2 f}{dx^2} + \left( \frac{1}{x} + \frac{2}{x+1} + \frac{2}{x-1} - \frac{1}{x+c} \right) \frac{df}{dx}
+ \left[ -\frac{2\nu}{x} + \frac{4\epsilon^2}{x^2} + \frac{D}{x+c} + \frac{A}{(x+1)} + \frac{A'}{(x-1)} + \frac{B}{(x+1)^2} + \frac{B'}{(x-1)^2} \right] f &= 0,
\end{align*}
\]
\[ f = 0, \quad (13) \]

where the notations are used

\[ A = \frac{-8\nu^2 + 35\epsilon^2 + 8\nu - 5M^2 + 8\nu/(c - 1)}{8}, \]
\[ A' = \frac{+8\nu^2 - 35\epsilon^2 + 8\nu + 5M^2 - 8\nu/(c + 1)}{8}, \]
\[ B = \frac{19\epsilon^2 - 8\nu^2 - 8\nu - 5M^2}{8}, B' = \frac{19\epsilon^2 - 8\nu^2 + 8\nu - 5M^2}{8}, D = -\frac{2\nu}{c^2 - 1}. \]

Restriction to massless case is reached by the change \( M \to 0, c \to \infty \). In (13) we have an equation with three regular singular points \( x = 0, -c, \infty \) and two irregular points \(-1, +1\) of the rank 2. Let us detail asymptotics near singular points (most interesting are physical points \( x = 0 \) and \( x = 1 \)):

\[ x \to 0, \quad f \sim x^\gamma, \quad \gamma = \pm 2i\epsilon; \]
\[ x \to +1, \quad f = (x - 1)^\alpha \exp\left(\frac{\beta}{x - 1}\right), \]
\[ \beta = \pm i\frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad \alpha = \pm i\frac{(\epsilon^2 - M^2) + M^2/2}{\sqrt{\epsilon^2 - M^2}}; \]
\[ x \to -1, \quad f = (x + 1)^{\alpha'} \exp\left(\frac{\beta'}{x + 1}\right), \]
\[ \beta' = \pm i\frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad \alpha' = \mp i\frac{(\epsilon^2 - M^2) + M^2/2}{\sqrt{\epsilon^2 - M^2}}; \]
\[ x \to -c, \quad f = (x + c)^\rho, \quad \rho = 0, 2, \]
\[ x \to \infty \quad (y = x^{-1}), \quad \left(\frac{d^2f}{dy^2} - \frac{2df}{ydy}\right)f = 0, \quad f(y) \sim \frac{1}{x^{\sigma}}, \quad \sigma = 0, 3. \]

Let us construct Frobenius solutions for eq. (13). It is convenient to introduce shortening notations \((\epsilon^2 - M^2)/4 = E, \epsilon^2 - M^2/2 = E'\), then eq. (13) is written as

\[ \frac{d^2f}{dx^2} + \left(\frac{1}{x} + \frac{2}{x + 1} + \frac{2}{x - 1} - \frac{1}{x + c}\right)\frac{df}{dx} + \left[-\frac{2\nu}{x} + \frac{4\epsilon^2}{x^2} + \frac{D}{x + c} + \frac{A}{(x + 1)^3} + \frac{A'}{(x - 1)^3} + \frac{B}{(x + 1)^4} + \frac{B'}{(x - 1)^4}\right]f = 0. \]

http://www.infim.ro/rrp
We construct Frobenius solutions with the use of the substitution
\[ f(x) = x^\gamma(x-1)^\alpha \exp\left(\frac{\beta}{x-1}\right) \exp(\frac{\beta'}{x+1}) F(x), \quad (20) \]
deriving the equation for \( F(x) \)
\[
\frac{d^2 F}{dx^2} + \left( \frac{1+2\gamma}{x} + \frac{2-2\alpha'}{x+1} + \frac{2+2\alpha}{x-1} - \frac{1}{x+c} - \frac{2\beta'}{(x+1)^2} - \frac{2\beta}{(x-1)^2} \right) \frac{dF}{dx} \\
+ \left[ \frac{3\beta - \beta' + 4\alpha + 2\alpha' - \beta' + 2\alpha\beta + 4\gamma + 4\gamma\alpha + 4\gamma\beta - \alpha\beta' + \beta\alpha' + 2A'}{2(x-1)} \right. \\
- \frac{\beta}{(1+c)^2(x-1)} - \frac{\alpha}{(1+c)(x-1)} + \frac{E + \beta^2}{(x-1)^4} + \frac{-E' - 2\alpha\beta}{(x-1)^3} \\
+ \frac{E' - 3\alpha'\beta'}{2} + \frac{1}{2} \frac{2B' + 2\alpha^2 - 4\beta + 2\alpha - 4\gamma\beta - 2\beta\alpha' + \beta\beta'}{(x-1)^2} \\
+ \frac{\beta\beta' - 4\alpha' - 2\alpha\beta - 4\gamma\alpha - \beta\alpha' + 3\beta' + \alpha\beta' + 4\gamma\beta' - \beta - 4\gamma - 2\alpha + 2A}{2(x+1)} \\
- \frac{\beta'}{(-1+c)^2(x+1)} - \frac{\alpha'}{(-1+c)(x+1)} + \frac{1}{c(c-1)^2(x+1)^2} [Dc^5 + (\gamma + \alpha + \alpha') c^4 + (\alpha' + \beta' - \alpha - 2D)c^3 \\
+ (-\alpha - 2\gamma + 2\beta' - 2\beta - \alpha')c^2 + (\alpha + \beta' + D - \alpha' + \beta) c + \gamma] \right] F = 0.
\]
Imposing needed constraints (here we have 8 variants)
\[
\gamma^2 + 4\epsilon^2 = 0 \implies \gamma = \pm 2i\epsilon; \\
-2\alpha\beta - E' = 0, \quad \beta^2 + E = 0 \implies \beta = \pm i\sqrt{E}, \quad \alpha = \pm i \frac{E'}{2\sqrt{E}}; \\
-2\alpha'\beta' + E' = 0, \quad \beta^2 + E = 0 \implies \beta' = \pm i\sqrt{E}, \quad \alpha' = \mp i \frac{E'}{2\sqrt{E}}; \quad (21)
\]
we arrive at the simpler equation for \( F(x) \) (we use its shortened form)
\[
F'' + \left( \frac{n}{x} + \frac{n_1}{x-1} + \frac{n_2}{(x-1)^2} + \frac{n_3}{x+1} + \frac{n_4}{(x+1)^2} + \frac{n_5}{x+c} \right) F' \\
+ \left( \frac{m}{x} + \frac{m_1}{x-1} + \frac{m_2}{(x-1)^2} + \frac{m_3}{x+1} + \frac{m_4}{(x+1)^2} + \frac{m_5}{x+c} \right) F = 0. \quad (22)
\]
Multiplying this by \(x(x+c)(x-1)^2(x+1)^2\), we get
\[
x^6 + cx^5 - 2x^4 - 2cx^3 + x^2 + cx \right] F'' + \left[ (n + n_1 + n_3 + n_5) x^5 + \left( (n + n_1 + n_3) c + n_1 + n_2 - n_3 + n_4 \right) x^4 + (n_1 + n_2 - n_3 + n_4) c - 2n - n_1 + 2n_2 - n_3 - 2n_4 - 2n_5 \right] x^3 + \left( -2n - n_1 + 2n_2 - n_3 - 2n_4 \right) c - n_1 + n_2 + n_3 + n_4 \right] x^2 + \left( (n_1 + n_2 + n_3 + n_4) c + n + n_5 \right) x + nc \right] F' + \left[ (m + m_1 + m_3 + m_5) x^5 + \left( (m + m_1 + m_3) c + m_1 + m_2 - m_3 + m_4 \right) x^4 + (m_1 + m_2 - m_3 - m_4) c - 2m - m_1 + 2m_2 - m_3 - 2m_4 - 2m_5 \right] x^3 + \left( -2m - m_1 + 2m_2 - m_3 - 2m_4 \right) c - m_1 + m_2 + m_3 + m_4 \right] x^2 + \left( (m_1 + m_2 + m_3 + m_4) c + m + m_5 \right) x + mc \right] F = 0.
\]
Solutions of the last equation may be searched as power series \(F = \sum_{k=0}^{\infty} b_k x^k\), further we produce 7-term recurrent relations:
\[
i = 0, \quad nb_1 + mb_0 = 0,
\]
\[
i = 1, \quad 2cb_2 + 2ncb_2 + \left[ (n_1 + n_2 + n_3 + n_4) c + n + n_5 \right] b_1 + mc b_1 + \left[ (n_1 + m_2 + m_3 + m_4) c + m + m_5 \right] b_0 = 0,
\]
\[
i = 2, \quad +3ncb_3 + 6cb_3 + 2b_2 + 2 \left[ (n_1 + n_2 + n_3 + n_4) c + n + n_5 \right] b_2 + mc b_2 + \left[ (n_1 + m_2 - m_3 + m_4) c - n_1 + n_2 + n_3 + n_4 \right] b_1 + \left[ (m_1 + m_2 + m_3 + m_4) c + m + m_5 \right] b_0 = 0,
\]
\[
i = 3, \quad 12cb_4 + 4ncb_4 + 6b_3 + 3 \left[ (n_1 + n_2 + n_3 + n_4) c + n + n_5 \right] b_3 + mc b_3 - 4cb_2 + 2 \left[ (n + n_1 + 2n_2 - n_3 - 2n_4) c - n_1 + n_2 + n_3 + n_4 \right] b_2 + \left[ (n_1 + m_2 + m_3 + m_4) c + m + m_5 \right] b_1 + \left[ (m_1 + m_2 - m_3 + m_4) c - 2n - n_1 + 2n_2 - n_3 - 2n_4 - 2n_5 \right] b_0 = 0,
\]
and the main recurrence relation reads
\[
i = 6, 7, 8..., \quad (m + m_1 + m_3 + m_5) b_{i-5} + \left[ (i-4)(i-5) + (n + n_1 + n_2 + n_4)(i-4) \right] b_{i-6} + \left[ (m + m_1 + m_3) c + m_1 + m_2 - m_3 + m_4 \right] b_{i-4} + \left[ (c(i-3)(i-4) + (n + n_1 + n_3)c + n_1 + n_2 - n_3 + n_4)(i-4) \right] b_{i-3} + \left[ (m_1 + m_2 - m_3 + m_4) c - 2n - m_1 + 2m_2 - m_3 - 2m_4 - 2m_5 \right] b_{i-3} + \left[ (m + m_1 + 2m_2 - m_3 - 2m_4) c - m_1 + m_2 + m_3 + m_4 \right] b_{i-2}.
In accordance with the Poincaré–Perrone method, we divide the main recurrent formula by $b_{i-5}$:

$$b_1 \frac{b_{i-1}}{b_{i-4}} \frac{b_{i-2}}{b_{i-3}} \frac{b_{i-1}}{b_{i-5}} \frac{b_{i-2}}{b_{i-3}} \frac{b_{i-1}}{b_{i-5}}$$

the roots are $R = 0$, $R = \pm 1$, $R = -1/c$. Therefore, possible convergence radii are $1, c > 1, \infty$. Note that the minimal convergence radius $R_{\text{conv}} = 1$ covers the whole physical region for the variable, $x \in (0, 1)$.
4. NUMERICAL STUDY

Let us list 8 solutions (note the change \((x - 1) < 0\) to \((1 - x) > 0\), it makes the involved functions single-valued in physical region \(x \in (0, 1)\))

\[
f(x) = x^{\gamma}(1 - x)^{\alpha} \exp\left(\frac{\beta}{1 - x}\right) (x + 1)^{\alpha'} \exp\left(\frac{\beta'}{x + 1}\right) F(x),
\]

where

\[
\begin{align*}
\beta &= \pm i\Gamma, \quad \alpha = \pm i\Sigma, \quad \beta' = \pm i\Gamma, \quad \alpha' = \pm i\Sigma, \\
\Gamma &= \frac{\sqrt{\epsilon^2 - M^2} - 2}{2}, \quad \Sigma = \frac{\epsilon^2 - M^2/2}{\sqrt{\epsilon^2 - M^2}}, \quad \gamma = \pm 2\epsilon.
\end{align*}
\]

In order to get the massless particle case, it suffices to make the formal changes in parameters:

\[
c \to \infty, \quad \Gamma = \frac{1}{2}\epsilon, \quad \Sigma = \epsilon.
\]

In the following we study only the massless case. Correspondingly, we have the following substitutions for 8 solutions (they are collected into pairs)

\[
\begin{align*}
g_1(x) &= h_1(x)e^{\frac{i\epsilon}{2(x - 1)}} + \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}, \\
g_2(x) &= h_2(x)e^{-\frac{i\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^{-i\epsilon} x^{2i\epsilon} (x + 1)^{i\epsilon}, \\
g_3(x) &= h_3(x)e^{-\frac{i\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^{-i\epsilon} x^{2i\epsilon} (x + 1)^{i\epsilon}, \\
g_4(x) &= h_4(x)e^{\frac{i\epsilon}{2(x - 1)}} + \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}, \\
g_5(x) &= h_5(x)e^{\frac{i\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}, \\
g_6(x) &= h_6(x)e^{\frac{i\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}, \\
g_7(x) &= h_7(x)e^{\frac{i\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}, \\
g_8(x) &= h_8(x)e^{i\frac{\epsilon}{2(x - 1)}} - \frac{i\epsilon}{2(x + 1)} (1 - x)^i\epsilon x^{2i\epsilon} (x + 1)^{-i\epsilon}.
\end{align*}
\]

We need explicit form of 2-nd order equations for all \(h_i(x)\), it suffices to write down only 4 cases, \(h_1, h_3, h_5, h_7\), because equations and solutions are divided in pairs of conjugate ones:

\[
\begin{align*}
&h''_1 + \left[\frac{4i\epsilon}{x} + \frac{2i\epsilon}{x - 1} - \frac{2i\epsilon}{x + 1} - \frac{i\epsilon}{(x - 1)^2} - \frac{i\epsilon}{(x + 1)^2} + \frac{2}{x - 1} + \frac{1}{x} + \frac{2}{x + 1}\right]h'_1 \\
&+ \left[-\frac{35\epsilon^2}{4(x - 1)} + \frac{12\epsilon^2}{x} - \frac{13\epsilon^2}{4(x + 1)} + \frac{11\epsilon^2}{4(x - 1)^2} - \frac{5\epsilon^2}{4(x + 1)^2} + \frac{11\epsilon}{2(x - 1)}\right]h_1 = 0.
\end{align*}
\]
Several typical graphs of series are given in Fig.4–Fig.5. Two binary parts of 4 converging series (they depend on quantum numbers $\epsilon, \nu$) demonstrate evident asymptotical behavior at $r \to \infty$.

Numerical study shows that real and imaginary parts of the series sums $R_i(r)$, $I_i(r)$ (i = 1, 3, 5, 7) will designate real and imaginary parts of 4 converging series (they depend on quantum numbers $\epsilon, \nu = j + 1/2$).

Several typical graphs of series are given in Fig.4–Fig.5.

Numerical study shows that real and imaginary parts of the series sums $R_i(r)$, $I_i(r)$ demonstrate evident asymptotical behavior at $r \to +\infty$.

Numerical study of the relevant Wronskians shows that all solutions are linearly independent.

Below symbols $R_i(r)$ and $I_i(r)$ (i = 1, 3, 5, 7) will designate real and imaginary parts of 4 converging series (they depend on quantum numbers $\epsilon, \nu = j + 1/2$).
Table 1.

Asymptotical behavior of the real part of the series

<table>
<thead>
<tr>
<th>$Re \ h_i$</th>
<th>$h_i(500)$</th>
<th>$h_i(750)$</th>
<th>$h_i(1000)$</th>
<th>$h_i(1250)$</th>
<th>$h_i(1500)$</th>
<th>$h_i(1000)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>1.0133</td>
<td>1.0043</td>
<td>1.0021</td>
<td>1.0012</td>
<td>1.0008</td>
<td>12092.5</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>1.0138</td>
<td>1.0045</td>
<td>1.0022</td>
<td>1.0013</td>
<td>1.0008</td>
<td>22583.1</td>
</tr>
<tr>
<td>$i = 5$</td>
<td>1.0170</td>
<td>1.0055</td>
<td>1.0027</td>
<td>1.0016</td>
<td>1.0011</td>
<td>1206.7</td>
</tr>
<tr>
<td>$i = 7$</td>
<td>1.0170</td>
<td>1.0055</td>
<td>1.0027</td>
<td>1.0016</td>
<td>1.0011</td>
<td>1206.7</td>
</tr>
</tbody>
</table>

Table 2.

Asymptotical behavior of the imaginary part of the series

<table>
<thead>
<tr>
<th>$Im \ h_i$</th>
<th>$h_i(500)$</th>
<th>$h_i(750)$</th>
<th>$h_i(1000)$</th>
<th>$h_i(1250)$</th>
<th>$h_i(1500)$</th>
<th>$h_i(1000)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>1.0349</td>
<td>1.0112</td>
<td>1.0055</td>
<td>1.0033</td>
<td>1.0021</td>
<td>279.989</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0.9290</td>
<td>0.9744</td>
<td>0.9868</td>
<td>0.9920</td>
<td>0.9946</td>
<td>284.854</td>
</tr>
<tr>
<td>$i = 5$</td>
<td>1.0132</td>
<td>1.0043</td>
<td>1.0021</td>
<td>1.0012</td>
<td>1.0000</td>
<td>-11203.6</td>
</tr>
<tr>
<td>$i = 7$</td>
<td>1.0132</td>
<td>1.0043</td>
<td>1.0021</td>
<td>1.0012</td>
<td>1.0009</td>
<td>-11203.6</td>
</tr>
</tbody>
</table>

5. TUNNELING PROCESS

Let us examine tunneling effect for the particle moving the the barrier from the right. To this end, we start with solutions $g_2(x)$, its asymptotic at $x \to 0 \ (r_* \to -\infty)$ is given by the formula

$$g_2(x) = e^{-2i\epsilon \ln x} e^{-i\epsilon} = e^{-i\epsilon} e^{-i\epsilon r_*}. \quad (31)$$

With notations

$$G_2(x) = e^{+i\epsilon r_*} g_2(x), \quad G_2(r \to -\infty) = e^{-i\epsilon r_*}, \quad (32)$$

we may formulate the following Cauchy problem

$$G_2(x), \quad \frac{d}{dx} G_2(x), \quad x_0 = 10^{-5}, \quad (r_*)_0 = -22. \quad (33)$$

The graphs for $G_2(r_*)$ are presented in Fig. 6 and Fig. 7. The tunneling process is described by the general formula

$$e^{-i\epsilon r_*} \iff A e^{-i\epsilon r_*} + B e^{+i\epsilon r_*}, \quad \text{or} \quad \frac{1}{A} e^{-i\epsilon r_*} \iff e^{-i\epsilon r_*} + \frac{B}{A} e^{+i\epsilon r_*}; \quad (34)$$

reflection and penetration coefficients are defined as

$$R = |\frac{B}{A}|^2, \quad D = |\frac{1}{A}|^2. \quad (35)$$

Let us consider two close points in the region $r_* \to +\infty$, $s_1 = \epsilon r_{1*}$ and $s_2 = \epsilon r_{2*}$.
\( e^{-i\epsilon s_1} A + e^{i\epsilon s_1} B = N_1, \quad e^{-i\epsilon s_2} A + e^{i\epsilon s_2} B = N_2. \) \( (36) \)

The values \( N_1 \) and \( N_2 \) are known from results of solving Cauchy problem, so there arises the linear system with respect to variables \( A \) and \( B \). Solving the system (36) numerically at several different values \( s_1 \) and \( s_2 \), we obtain

**Table 3.**

Coefficients \( |D| \) and \( |R| \) at \( \epsilon = 1 \)

| \( s_1 \) | \( s_2 \) | \( |D| \)         | \( |R| \)             |
|-------|-------|----------------|----------------------|
| 100   | 101   | \( 2.587117 \times 10^{-9} \) | 0.999999999999330    |
| 200   | 201   | \( 2.587702 \times 10^{-9} \) | 0.999999999999330    |
| 300   | 301   | \( 2.588085 \times 10^{-9} \) | 0.999999999999330    |
| 400   | 401   | \( 2.588248 \times 10^{-9} \) | 0.999999999999330    |
| 500   | 501   | \( 2.588268 \times 10^{-9} \) | 0.999999999999330    |
| 600   | 601   | \( 2.588230 \times 10^{-9} \) | 0.999999999999330    |

We can see (Table 3) that different choices of \( s_1 \) and \( s_2 \) lead to the very close results. We have calculated coefficients \( D \) and \( R \) at different values of energy (at fixed \( s_1 = 100 \) and \( s_2 = 101 \)):

**Table 4.**

Coefficients \( D \) and \( R \) at different values of energy

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( D )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2.5880 \times 10^{-9} )</td>
<td>0.9999999999999999</td>
</tr>
<tr>
<td>3/2</td>
<td>( 1.7582 \times 10^{-7} )</td>
<td>0.9999999999960</td>
</tr>
<tr>
<td>2</td>
<td>( 6.5645 \times 10^{-6} )</td>
<td>0.99999569020</td>
</tr>
<tr>
<td>5/2</td>
<td>( 9.8942 \times 10^{-6} )</td>
<td>0.99999020869</td>
</tr>
<tr>
<td>3</td>
<td>( 9.9975 \times 10^{-6} )</td>
<td>0.99989999925</td>
</tr>
<tr>
<td>7/2</td>
<td>( 10.0000 \times 10^{-6} )</td>
<td>0.99989999954</td>
</tr>
</tbody>
</table>

These values (see Table 4) are illustrated in Fig. 8.

In similar manner, we could examine tunneling effect when the particle moves to the barrier from the left. This inverse tunneling effect effectively should contribute to evaporation of the black hole.

**6. CONCLUSIONS**

For Dirac particle, the analytical and numerical study of the tunneling process through potential barrier generated by Schwarzschild black hole metric is done. We
construct solutions in explicit form and prove that the involved power series are converged in the whole physical region of the radial variable $r \in (1, +\infty)$. The tunneling effect is studied for situation when the particle falls on the barrier from outside. Mathematical structure of the derived relations is exact, however analytical expressions for the sums of involved convergent powers series are not known, so our further study was based on numerical summing the series and asymptotical behavior of the complete solutions. The calculations are implemented with the use of the Mathematica system.

REFERENCES

8. D.V. Gal’tsov, *“Particles and fields in vicinity of black holes”* (Moscow State University, Moscow, 1986)
the background of Lobachevsky geometry: analytical and numerical study, visualization, Stud. i Mater. EU in Warsaw **10**, **45** (2015)


FIGURES

![Figure 1](image1.png)  
**Fig. 1** – The function \( r = r(r_*) \)

![Figure 2](image2.png)  
**Fig. 2** – The function \( r_* = r_*(r) \)

![Figure 3](image3.png)  
**Fig. 3** – Potential function \( \phi(r_*) \)
Fig. 4 – The graph of the series $h_1(r_\ast)$, $\epsilon = 1$, $\nu = 5$

Fig. 5 – The graph of the series $h_3(r_\ast)$, $\epsilon = 1$, $\nu = 5$

Fig. 6 – The graph of $\text{Re} \ G_2(r_\ast)$
Fig. 7 – The graph of $\text{Im} \ G_2(r_*)$

Fig. 8 – Penetration and reflection coefficients