THE MULTIPLE BRIGHT SOLITON PAIRS OF THE FULLY $\mathcal{PT}$-SYMMETRIC NONLOCAL DAVEY–STEWARDSON I EQUATION

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Abstract. This article investigates the dynamics of multiple bright soliton pair interactions in the fully $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson I equation. The bright soliton pair solutions are derived by employing the bilinear KP-hierarchy reduction method, and are expressed in terms of determinants. To study the interactions of the multiple soliton pairs, the long-time asymptotic analysis for these soliton solutions is performed by using the analysis of determinants, and the asymptotic expressions of the $N$ individual soliton pair solutions are given as the sum of expressions for the $2N$ single soliton solutions. The asymptotics shows that the soliton pairs only exhibit elastic collisions and the two solitons in each soliton pair share equal amplitudes.

Key words: Fully $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson I equation, Bright solitons, Long-time asymptotic analysis, KP-hierarchy reduction method.

1. INTRODUCTION

The Davey–Stewartson equation is a fascinating integrable model that has a remarkable mathematical structure [1]. This system is derived from the non-integrable Benney–Roskes–Davey–Stewartson system, which is commonly employed to describe the evolution of a two–dimensional water wavepacket [2]. It is classified into two distinct types known as the Davey–Stewartson I equation and the Davey–Stewartson II equation. This classification is based on the strength of the surface tension, with the Davey–Stewartson I equation corresponding to strong surface tension and the Davey–Stewartson II equation to weak surface tension [3]. It is known that both types of Davey–Stewartson equations can provide the multidimensional version of the classical nonlinear Schrödinger equation [4]. Additionally, this remarkable model can also be established from the self-dual Yang–Mills equation through a suitable transformation [5]. In subsequent studies, researchers have extended the application of this equation from water waves to various other domains, encompassing fluid dynamics [3], nonlinear optics [6, 7], plasma physics [8], and lattice dynamics [9].
In 2016, Fokas [10] introduced several nonlocal versions of the Davey–Stewartson equation with $\mathcal{PT}$-symmetry, termed as nonlocal Davey–Stewartson equations. These nonlocal Davey–Stewartson equations can provide two-dimensional analogues of the integrable nonlocal nonlinear Schrödinger equation introduced by Ablowitz and Musslimani [11]. After the work of Fokas [10], Ablowitz and Musslimani [12] also paid a considerable attention for these nonlocal Davey–Stewartson equations. We point out that these nonlocal Davey–Stewartson equations are also completely integrable as well as the two types of classical Davey–Stewartson equations. They also admit Lax pairs and were solved by Darboux transformation method [13–16]. For the $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson I equation, different types of soliton solutions on a nonzero background [17] and the general lump-soliton solutions on a background of periodic line waves [18] were constructed by using the Hirota bilinear method and the KP-hierarchy reduction technique. Those solutions were expressed in terms of determinants of Gramm type. We also point out that the rational and semi-rational solutions of the nonlocal Davey–Stewartson equations were studied by Rao et al. [19] and line rogue waves of the nonlocal Davey–Stewartson I equation were investigated in another paper by Rao et al. [20]. In contrast with the line rogue waves of the local Davey–Stewartson equation derived by Ohta and Yang [21, 22], the line rogue waves of the nonlocal Davey–Stewartson I equation can attain much higher amplitudes. It is known that the fundamental line rogue waves of the local Davey–Stewartson I equation can acquire maximum amplitudes as three times the background amplitude [21, 23–26]. However, the maximum amplitudes of the fundamental line rogue waves of the nonlocal Davey–Stewartson I equation can tend to infinity under suitable parameter choices.

Another distinctive feature of the nonlocal Davey–Stewartson equations is that they exhibit a much richer variety of solitons, such as lump solutions on a background of periodic line waves [19,20]. We also mention that bright solitons on the zero and periodic wave background in the space-shifted $\mathcal{PT}$-symmetric nonlocal nonlinear Schrödinger equation were also investigated [27]. Families of different types of solitons in many physical contexts were investigated in a series of recent works; see, for example, Refs. [28–37].

The fully $\mathcal{PT}$-symmetric nonlocal Davey–Stewartson I equation is written as follows [10, 12]

\begin{align}
\dot{A} & = A_{xx} + A_{yy} + (\epsilon A \hat{A} - 2Q)A, \\
Q_{xx} - Q_{yy} & = (\epsilon A \hat{A})_{xx}, \epsilon = \pm 1,
\end{align}

where $A = A(x,y,t), Q = Q(x,y,t)$, and the hat symbol ‘$\hat{}$’ denotes the nonlocal symmetry of a function, defined as $\hat{A}(x,y,t) = A(-x,-y,t)$, with the asterisk * indicating the complex conjugation. The different types of solitons of this equation have been investigated [1]. Very recently, a generalized version of Eq. (1), termed as
the space-shifted nonlocal Davey–Stewartson I equation, was considered [38], and its multiple dark and antidark soliton pair solutions were derived by the bilinear KP-hierarchy reduction method. The long-time asymptotics of the soliton pair solutions were given to indicate the interaction properties of the multiple dark and antidark soliton pairs. However, the long-time asymptotics of the bright soliton pair solutions on vanishing boundary condition have not been studied, to the best of our knowledge.

In this article we investigate the multiple bright soliton pair solutions on vanishing boundary conditions and their interaction dynamics in the framework of the fully $PT$-symmetric nonlocal Davey–Stewartson I equation (1). The paper is organized as follows. In Sec. 2, we first present the multiple bright soliton pair solutions on vanishing boundary conditions to Eq. (1) in the form of Theorem 1, and then investigate the interaction dynamics of these bright soliton pairs through long-time asymptotics. In Sec. 3, we give the detailed derivations of the multiple bright soliton pair solutions presented in Theorem 1. We conclude this work in Sec. 4.

2. THE MULTIPLE BRIGHT SOLITON PAIR SOLUTIONS AND THEIR DYNAMICS

In this Section we first present the multiple bright soliton pair solutions on vanishing boundary condition $(A, Q) \rightarrow (0, 0)$ for the fully $PT$-symmetric nonlocal Davey–Stewartson I equation (1), and then we study their collision properties via the asymptotic analysis.

2.1. THE MULTIPLE BRIGHT SOLITON PAIR SOLUTIONS IN DETERMINANT FORMS

The multiple bright soliton pair solutions in terms of determinants to the fully $PT$-symmetric nonlocal Davey–Stewartson I equation (1) can be presented by the following Theorem, which is proved in Sec. 3.

**Theorem 1.** The fully $PT$-symmetric nonlocal Davey–Stewartson I equation (1) admits the following multiple soliton pair solutions on vanishing boundary conditions

$$A = \sqrt{2g} f, Q = -2(\ln f)_{xx},$$

where

$$f(x, y, t) = \left| M \right|, g(x, y, t) = \begin{vmatrix} M \Phi' \\ -\Psi \\ 0 \end{vmatrix},$$

with

$$M = \text{mat}_{1 \leq i, j \leq 2N} (m_{i,j}), \Phi = (\Phi_1, \Phi_2, \cdots, \Phi_2N), \Psi = (\Psi_1, \Psi_2, \cdots, \Psi_{2N}).$$
and
\[ m_{i,j} = e^{\xi_i + \zeta_i} \left[ \frac{1}{p_i + p_j} + \frac{1}{q_i + q_j} \right], \]
\[ \Phi_i = e^{\xi_i}, \Psi_j = 1, \zeta_i = \xi_i - \eta_i, \]
\[ \xi_i = \frac{1}{2} p_i x + \frac{1}{2} p_i y - \frac{i}{2} p_i^2 t + \xi_i,0, \]
\[ \eta_i = -\frac{e}{2} q_i x + \frac{e}{2} q_i y - \frac{i}{2} q_i^2 t + \eta_i,0. \]

Here the complex parameters \( p_i, q_j, \xi_i,0, \eta_j,0 \) have to satisfy the following restrictions
\[ p_{N+i} = -p_i, q_{N+j} = -q_j, \xi_{N+i,0} = \xi_{i,0}, \eta_{N+j,0} = \eta_{j,0}, \]
for \( i, j = 1, 2, \ldots, N \).

**Remark 1.** As \( x, y, t \to \pm \infty, (A(x, y, t), Q(x, y, t)) \to (0, 0) \), thus, the solutions are on vanishing boundary conditions.

**Remark 2.** Due to the fact that the matrix elements \( m_{i,j} \) admit the property \( m_{i,j}^* = m_{j,i} \), thus the function \( f \) is real. Consequently, this indicates that the solution \( Q(x, y, t) = -2(\ln f)_{xx} \) is also real. This is different from the multiple dark and antidark soliton pairs in fully \( P \, T \)-symmetric nonlocal Davey–Stewartson I equation, which is complex [38].

### 2.2. Dynamics of the Single Bright Soliton Pair Interactions

The single bright soliton pair solutions to Eq. (1) are derived from Theorem 1 by taking \( N = 1 \) in formulae (2)–(5). The functions \( f(x, y, t), g(x, y, t) \) of the single soliton pair solutions (2) can be rewritten in the following determinant forms
\[ f(x, y, t) = \begin{vmatrix} \tilde{m}_{1,1} & \tilde{m}_{1,2} \\ \tilde{m}_{2,1} & \tilde{m}_{2,2} \end{vmatrix}, \]
\[ g(x, y, t) = \begin{vmatrix} \tilde{m}_{1,1} & \tilde{m}_{1,2} & e^{\zeta_1} \\ \tilde{m}_{2,1} & \tilde{m}_{2,2} & e^{\zeta_2} \end{vmatrix}, \]
where
\[ \tilde{m}_{1,1} = e^{\xi_1 + \zeta_1} \left[ \frac{1}{p_1 + p_1} + \frac{1}{q_1 + q_1} \right], \tilde{m}_{1,2} = e^{\xi_1 + \zeta_2} \left[ \frac{1}{p_1 - p_1} + \frac{1}{q_1 - q_1} \right], \]
\[ \tilde{m}_{2,1} = -\tilde{m}_{1,2}(-x, -y, t), \tilde{m}_{2,2} = -\tilde{m}_{1,1}(-x, -y, t), \]
and
\[ \zeta_1 = \frac{1}{2} (p_1 + \epsilon q_1) x + \frac{1}{2} (p_1 - \epsilon q_1) y - \frac{i}{2} (p_1^2 - q_1^2) t + \xi_{1,0} - \eta_{1,0}. \]

To keep the solutions being non-singular, we have to restrict \( p_1, Rq_1, R > 0 \). Hereafter, the subscripts \( R \) and \( I \) represent the real and imaginary parts of a function or a parameter, respectively. Figure 1 shows the single bright soliton pair solutions at times \( t = -2, 0, 2 \), illustrating that the two bright solitons move along two parallel trajectories in the \((x, y)\)-plane, but in opposite directions.
Consequently, the asymptotic expressions of the two bright solitons are as follows:

\[ \zeta_0 + \zeta_1^* = z_0^0 x + z_0^0 y + z_0^0 t + z_0, \]  

where \( z_0^0 = \partial_x (\zeta_0 + \zeta_1^*) , \partial_y (\zeta_0 + \zeta_1^*) , \partial_t (\zeta_0 + \zeta_1^*) \) and \( z_0 = \zeta_{1,0} - \eta_{1,0} + \xi_{1,0} - \eta_{1,0}^* \). Then, \( (\zeta_1 + \zeta_1^*) (-x, -y, t) \) can be represented as

\[ (\zeta_1 + \zeta_1^*) (-x, -y, t) = -z_0^0 x - z_0^0 y + z_0^0 t + z_0. \]

In the following asymptotic analysis, we assume \( z_x, z_y, z_t > 0 \) without loss of generality. Then we can derive that \( (\zeta_1 + \zeta_1^*) (-x, -y, t) \to \pm \infty \) as \( t \to \pm \infty \) when \( \zeta_1 + \zeta_1^* \approx 0(1) \), and \( \zeta_1 + \zeta_1^* \to \pm \infty \) as \( t \to \pm \infty \) when \( (\zeta_1 + \zeta_1^*) (-x, -y, t) \approx 0(1) \). Consequently, the asymptotic expressions of the two bright solitons are as follows:

(a) Before collision \( (t \to -\infty) \)

\[ \text{Soliton } S_1^0 \]

\[ (\zeta_1 + \zeta_1^*) \approx 0, (\zeta_1 + \zeta_1^*) (-x, -y, t) \to -\infty) : \]

\[ A_1^{0-} S_1^0 \approx A_1^{0-} e^{\zeta_1 + \zeta_1^*} \text{ sech } [\zeta_1 + \zeta_1^*], \]

\[ Q_1^{0-} S_1^0 \approx Q_0 \text{ sech } ^2 [\zeta_1 + \zeta_1^*], \]

where

\[ A_1^{0-} = \frac{\sqrt{2}q_1^*}{(q_1 - q_1^*)^2} \sqrt[3]{\frac{(p_1 + p_1^*)^2 (q_1 - q_1^*)^2}{4q_1 q_1^*} }, \]

\[ \zeta_1 = \frac{1}{2} \ln \left[ -\frac{(q_1 + q_1^*)^2}{4q_1 q_1^* (p_1 + p_1^*)} \right], Q_0 = \frac{(z_0^0)^2}{8}. \]
Soliton $S_{1}^{[2]} ((\zeta_2 + \zeta_2^*)(-x, -y, t) \approx 0, \zeta_1 + \zeta_1^* \rightarrow -\infty)$:

$$A_{S_{1}^{[2]}}^{-} \approx -A_{1}^{(0)-} e^{i \zeta_1} \text{sech} \left[ \zeta_{1,R}(-x, -y, t) + \theta_1^- \right],$$
$$Q_{S_{1}^{[2]}}^{-} \approx Q_0 \text{sech}^2 \left[ \zeta_{1,R}(-x, -y, t) + \theta_1^- \right]. \quad (14)$$

(b) After collision ($t \rightarrow +\infty$)

Soliton $S_{1}^{[1]} (\zeta_1 + \zeta_1^* \approx 0, (\zeta_1 + \zeta_1^*)(-x, -y, t) \rightarrow +\infty)$:

$$A_{S_{1}^{[1]}}^{+} \approx A_{1}^{(0)+} e^{i \zeta_1} \text{sech} \left[ \zeta_{1,R} + \theta_1^+ \right],$$
$$Q_{S_{1}^{[1]}}^{+} \approx Q_0 \text{sech}^2 \left[ \zeta_{1,R} + \theta_1^+ \right], \quad (15)$$

where

$$A_{1}^{(0)+} = \frac{\sqrt{2}p_1}{(p_1 - p_1^*) (p_1 + p_1^*)} \sqrt{\frac{(q_1 + q_1^*) (p_1 - p_1^*)^2 (p_1 + p_1^*)^3}{4p_1^2}},$$
$$\theta_1^+ = \frac{1}{2} \ln \left[ -\frac{4p_1^* q_1 (q_1 + q_1^*)}{(p_1 + p_1^*) (p_1 - p_1^*)^2} \right]. \quad (16)$$

Soliton $S_{1}^{[2]} ((\zeta_2 + \zeta_2^*)(-x, -y, t) \approx 0, \zeta_1 + \zeta_1^* \rightarrow +\infty)$:

$$A_{S_{1}^{[2]}}^{+} \approx -A_{1}^{(0)+} e^{i \zeta_1} \text{sech} \left[ \zeta_{1,R}(-x, -y, t) + \theta_1^+ \right],$$
$$Q_{S_{1}^{[2]}}^{+} \approx Q_0 \text{sech}^2 \left[ \zeta_{1,R}(-x, -y, t) + \theta_1^+ \right]. \quad (17)$$

The asymptotic expressions of the two solitons in Eqs. (12)–(17) indicate that

$$|A_{S_{1}^{[1]}}^{+} (\zeta_{1,R})| = |A_{S_{1}^{[1]}}^{-} (\zeta_{1,R} - \theta_1^- + \theta_1^+)|$$

and

$$|A_{S_{1}^{[2]}}^{+} (\zeta_{1,R}(-x, -y, t))| = |A_{S_{1}^{[2]}}^{-} (\zeta_{1,R}(-x, -y, t) - \theta_1^- + \theta_1^+)|.$$ 

This reflects that the two bright solitons undergo elastic collisions, namely, their velocities, amplitudes, and shapes remain unaltered after collisions, except for the finite phase shift $\theta_1^+ - \theta_1^-$. Additionally, since the maximum amplitudes of the two solitons are $|A_{1}^{(0)+}|$ and $|A_{1}^{(0)-}|$, and because $|A_{1}^{(0)+}| = |A_{1}^{(0)-}|$ they share equal maximum amplitudes. Finally, the asymptotics of the single soliton pair solutions given by Eq. (7) are

$$A^\pm = A_{S_{1}^{[1]}}^\pm + A_{S_{1}^{[2]}}^\pm,$$
$$Q^\pm = Q_{S_{1}^{[1]}}^\pm + Q_{S_{1}^{[2]}}^\pm. \quad (18)$$
The comparison between the exact single soliton pair solutions and their asymptotics is illustrated in Fig. 2, where the two closely coincide.

![Comparison of exact single soliton pair solutions and asymptotics](image)

**Fig. 2** – (Colour online) The comparison of the exact single soliton pair solutions in Eq. (7) (blue solid line) with their asymptotics \((A^-, Q^-)\) outlined by Eq. (18) at time \(t = -3\) (red dotted line) along \(y = 0\) with the same parameters as in Fig. 1.

### 2.3. Dynamics of the Multiple Bright Soliton Pair Interactions

In this Section we study the dynamics of the multiple bright soliton pair interactions. To this end, we perform the asymptotic analysis as \(t \to \pm \infty\) for the soliton pair solutions (2) with \(N \geq 2\).

The solutions (2) are \(N\) bright soliton pairs. The two bright solitons in each soliton pair propagate along \(\zeta_i + \zeta_i^* \approx 0\) and \((\zeta_i + \zeta_i^*)(-x, -y, t) \approx 0\) for \(i = 1, 2, \ldots, N\). For convenience, they are denoted as soliton \(S_i^{[1]}\) and \(S_i^{[2]}\), respectively, and \(\zeta_i + \zeta_i^*\) are represented as follows

\[
\zeta_i + \zeta_i^* = z_{i,x}x + z_{i,y}y + z_{i,t}t + z_{i,0},
\]

where

\[
\begin{align*}
z_{i,x} &= \frac{\partial}{\partial x} (\zeta_i + \zeta_i^*), \\
z_{i,y} &= \frac{\partial}{\partial y} (\zeta_i + \zeta_i^*), \\
z_{i,t} &= \frac{\partial}{\partial t} (\zeta_i + \zeta_i^*), \\
z_{i,0} &= \zeta_{i,0} + \zeta_{i,0}.
\end{align*}
\]

Below, we give the detailed derivation for the asymptotics of the functions \(f\) and \(g\) in Eq. (3) along \(\zeta_i + \zeta_i^* \approx 0(1)\) as \(t \to -\infty\). To this end, we assume \(z_{i,x}, z_{i,y}, z_{i,t} > 0\) and \(\frac{z_{1,x}}{z_{1,x}} < \frac{z_{2,x}}{z_{2,x}} < \cdots < \frac{z_{N,x}}{z_{N,x}}\) without loss of generality.

Then, as \(t \to -\infty\), \(\zeta_j + \zeta_j^* \to +\infty\) when \(j < s\) and \(\zeta_j + \zeta_j^* \to -\infty\) when \(s < j \leq 2N\), along \(\zeta_s + \zeta_s^* \approx 0(1)\).
Therefore, the asymptotics of function $f$ is

\[
f_{s_{[\ell]}}^{-} = \frac{1}{p_{1} + p_{1}} \frac{1}{p_{1} + p_{2}} \cdots \frac{1}{p_{1} + p_{L-1}} \frac{\epsilon_{\xi}^{\ell}}{\epsilon_{\xi}^{\ell}} + \frac{1}{p_{2} + p_{1}} \frac{1}{p_{2} + p_{2}} \cdots \frac{1}{p_{2} + p_{L-1}} \frac{\epsilon_{\xi}^{\ell}}{\epsilon_{\xi}^{\ell}} + \cdots + \frac{1}{p_{L-1} + p_{1}} \frac{1}{p_{L-1} + p_{2}} \cdots \frac{1}{p_{L-1} + p_{L-1}} \frac{\epsilon_{\xi}^{\ell}}{\epsilon_{\xi}^{\ell}}
\]

\[
e^{\xi + \xi^*} D_{[\ell]}^{[0]}(p) D_{[\ell+1]}^{[2N]}(q) + D_{[\ell-1]}^{[0]}(p) D_{[\ell]}^{[2N]}(q),
\]

where

\[
D_{[\mu]}^{[k]} (z) = \frac{\prod_{i < j \leq k} (z_j - z_i)(z_j^* - z_i^*)}{\prod_{i, j \geq \mu} (z_j + z_j^*)}.
\]

(21)
The multiple bright soliton pairs of the fully $\mathcal{PT}$-symmetric nonlocal Davey--

The asymptotics of function $g$ is

\[
g_{S_{\ell}^{[1]}} = \frac{1}{p_{1}+p_{1}^*} \quad \frac{1}{p_{1}+p_{2}} \quad \cdots \quad \frac{1}{p_{1}+p_{2}^{[-1]}} \quad \frac{e^{\zeta_{\ell}^*}}{p_{1}+p_{2}^{[-1]}} \quad 0 \quad \cdots \quad 0 \quad 1
\]

\[
e^{\zeta_{\ell}} \left( \prod_{i=1}^{\ell-1} \frac{p_{i+1} - p_{i}}{p_{i} + p_{i}^{*}} \right) \left( \prod_{i=\ell+1}^{2N} \frac{q_{i}^{L} - q_{i}^{R}}{q_{i}^{L} + q_{i}^{R}} \right) D_{1}^{[-1]}(p) D_{2N}^{[-1]}(q).
\]

\[
(23)
\]

In a similar manner, the asymptotics of the functions $f$ and $g$ along $\zeta_{\ell} + \zeta_{\ell}^* \approx 0$ as $t \rightarrow +\infty$ and $(\zeta_{\ell} + \zeta_{\ell}^*)(x, y, t) \approx 0$ as $t \rightarrow \pm \infty$ can also be derived. Based on the asymptotics of functions $f$ and $g$, the asymptotics of the multiple soliton pair solutions (2) along all $\zeta_{\ell} + \zeta_{\ell}^* \approx 0$ and $(\zeta_{\ell} + \zeta_{\ell}^*)(x, -y, t) \approx 0$ as $t \rightarrow \pm \infty$ can be provided in the following explicit form:

(a) Before collision ($t \rightarrow -\infty$)

Soliton $S_{\ell}^{[1]} (\zeta_{\ell} + \zeta_{\ell}^* \approx 0)$:

\[
A_{S_{\ell}^{[1]}} A_{\ell}^{-} e^{\zeta_{\ell}^*} \text{sech} \left[ \zeta_{\ell,R} + \theta_{\ell}^{(1)} \right],
\]

\[
Q_{S_{y}} \approx Q_{\ell}^{[0]} \text{sech}^{2} \left[ \zeta_{\ell,R} + \theta_{\ell}^{(1)} \right],
\]

(24)
where

\[
A^{(1)}_\ell = \frac{\sqrt{2}}{2} \left( \prod_{i=1}^{\ell-1} \frac{p_i - p_i^*}{p_i + p_i^*} \right) \left( \prod_{i=\ell+1}^{2N} \frac{q_i^2 - q_i^*}{q_i^2 + q_i^*} \right) \left( \frac{D^1_{\ell-1}[p]}{D^1_{\ell+1}[p]} \frac{D^{2N}_{\ell-1}[q]}{D^{2N}_{\ell+1}[q]} \right)^{\frac{1}{2}},
\]

\[
\theta^{(1)}_\ell = \frac{1}{2} \ln \left( \frac{D^1_{\ell-1}[p]}{D^1_{\ell+1}[p]} \frac{D^{2N}_{\ell-1}[q]}{D^{2N}_{\ell+1}[q]} \right), \quad Q^{(0)}_\ell = \frac{(\zeta_{\ell,x})^2}{8}.
\]

Soliton \( S^{(2)}_{\ell} \) \((\zeta + \zeta^*)_{(-x, -y, t)} \approx 0) :

\[
A^{-}_{S[\ell]} \simeq A^{(2)-}_{\ell} e^{\zeta, t} \ \text{sech} \left[ \zeta_{\ell,R}(-x, -y, t) + \theta^{(2)-}_{\ell} \right],
\]

\[
Q^{-}_{S\ell} \simeq Q^{(0)}_{\ell} \ \text{sech}^2 \left[ \zeta_{\ell,R}(-x, -y, t) + \theta^{(2)-}_{\ell} \right],
\]

where

\[
A^{(2)-}_{\ell} = \frac{\sqrt{2}}{2} \left( \prod_{i=N+1}^{N+\ell-1} \frac{p_i - p_i^*}{p_i + p_i^*} \right) \left( \prod_{i=1}^{2N-\ell} \frac{q_i^2 - q_i^*}{q_i^2 + q_i^*} \right) \left( \frac{D^{N+\ell-1}_{N+1}[p]}{D^{N+\ell-1}_{N+1}[p]} \frac{D^{2N-\ell}[q]}{D^{2N-\ell}[q]} \right)^{\frac{1}{2}},
\]

\[
\theta^{(2)-}_{\ell} = \frac{1}{2} \ln \left( \frac{D^{N+\ell-1}_{N+1}[p]}{D^{N+\ell-1}_{N+1}[p]} \frac{D^{2N-\ell}[q]}{D^{2N-\ell}[q]} \right),
\]

and

\[
\tilde{p}^{[1]}_i = \begin{cases} p_i & \text{for } i = 1, 2, \ldots, N, \\ p_i+\ell-1 & \text{for } i = N + 1, N + 2, \ldots, 2N - \ell + 1, \end{cases}
\]

\[
\tilde{p}^{[2]}_i = \begin{cases} p_i & \text{for } i = 1, 2, \ldots, N, \\ p_i+\ell & \text{for } i = N + 1, N + 2, \ldots, 2N - \ell, \end{cases}
\]

\[
\tilde{q}^{[1]}_i = \begin{cases} q_i & \text{for } i = 1, 2, \ldots, N, \\ q_i+\ell-1 & \text{for } i = N + 1, N + 2, \ldots, 2N - \ell + 1, \end{cases}
\]

\[
\tilde{q}^{[2]}_i = \begin{cases} q_i & \text{for } i = 1, 2, \ldots, N, \\ q_i+\ell & \text{for } i = N + 1, N + 2, \ldots, 2N - \ell. \end{cases}
\]

(b) After collision \((t \to +\infty)\)

Soliton \( S^{(2)}_{\ell} \) \((\zeta_1 + \zeta_1^* \approx 0)\):

\[
A^{+}_{S[\ell]} \simeq A^{(1)+}_{\ell} e^{\zeta, t} \ \text{sech} \left[ \zeta_{\ell,R} + \theta^{(1)+}_{\ell} \right],
\]

\[
Q^{+}_{S\ell} \simeq Q^{(0)}_{\ell} \ \text{sech}^2 \left[ \zeta_{\ell,R} + \theta^{(1)+}_{\ell} \right],
\]
where

\[ A^{(1)+}_\ell = \frac{\sqrt{2}}{2} \left( \prod_{i=1}^{2N} \frac{p_i - p_{\ell}}{p_i + p_{\ell}} \right) \left( \prod_{i=1}^{\ell-1} \frac{q_i^* - q_i}{q_i + q_i^*} \right) \left( D_{\ell+1}^{[2N]}(p) D_1^{[\ell-1]}(q) \right)^{\frac{1}{2}}, \]

\[ \theta^{(1)+}_\ell = \frac{1}{2} \ln \left( \frac{D_{\ell+1}^{[2N]}(p) D_1^{[\ell-1]}(q)}{D_1^{[2N]}(p) D_1^{[\ell]}(q)} \right), \]  

Soliton \( S_\ell^{[2]} \) \((\zeta_\ell + \zeta_\ell^*) (\xi_x, -\xi_y, t) \approx 0:\)

\[ A^{(2)+}_{S_\ell^{[2]}} = A^{(2)+}_{\ell} \exp \left[ \frac{\theta^{(2)+}_\ell}{2} \right], \]

\[ Q_{S_\ell^{[2]}} \equiv Q_{S_\ell} \exp \left[ \frac{\theta^{(2)+}_\ell}{2} \right], \]  

where

\[ A^{(2)+}_\ell = \frac{\sqrt{2}}{2} \left( \prod_{i=1}^{2N-\ell} \frac{p_{N+i-\ell} - p_i}{p_{N+i-\ell} + p_i} \right) \left( \prod_{i=N+1}^{\ell} \frac{q_{N+i-\ell} - q_i}{q_i + q_{N+i-\ell}} \right) \left( D_1^{[2N-\ell]}(p^{[2]}) D_1^{[N+\ell-1]}(q) \right)^{\frac{1}{2}}, \]

\[ \theta^{(2)+}_\ell = \frac{1}{2} \ln \left( \frac{D_1^{[2N-\ell]}(p^{[2]}) D_1^{[N+\ell-1]}(q)}{D_1^{[2N]}(p^{[2]}) D_1^{[N+\ell]}(q)} \right). \]

Based on the above algebraic calculations, we represent the asymptotic expression of \( N \) bright soliton pair solutions (2) as the sum of \( 2N \) single bright soliton solutions through the following Theorem.

**Theorem 2.** As \( t \to \pm \infty \), the asymptotics of the \( N \) bright soliton pair solutions (2) are

\[ A^\pm = \sum_{\ell=1}^{N} \left( A^{\pm}_{S_\ell^{[1]}} + A^{\pm}_{S_\ell^{[2]}} \right), \]

\[ Q^\pm = \sum_{\ell=1}^{N} \left( Q^{\pm}_{S_\ell^{[1]}} + Q^{\pm}_{S_\ell^{[2]}} \right), \]  

where \( A^{\pm}_{S_\ell^{[j]}} \) and \( Q^{\pm}_{S_\ell^{[j]}} \) for \( j = 1, 2 \) and \( \ell = 1, 2, \ldots, N \) are given by Eqs. (24), (26), (29), and (31).

The quantities \( A^{\pm}_{S_\ell^{[j]}} \) obey the relation \(|A^{\pm}_{S_\ell^{[1]}}| = |A^{\pm}_{S_\ell^{[2]}}|\). This indicates that the two solitons in each soliton pair have equal amplitudes and remain unaltered after collision. The centers of the two solitons in each soliton pair alter from \( \zeta_{\ell,R}(x,y,t) + \theta^{(1)-}_\ell \) and \( \zeta_{\ell,R}(-x,-y,t) + \theta^{(2)-}_\ell \) to \( \zeta_{\ell,R}(x,y,t) + \theta^{(1)+}_\ell \) and \( \zeta_{\ell,R}(-x,-y,t) + \theta^{(2)+}_\ell \) as \( t \to -\infty \) to \( t \to +\infty \), thus the velocities of the two solitons also remain the same after collision. The solitons \( S_\ell^{[j]} \) in a soliton pair experience the phase shifts \( \theta^{(1)+}_\ell \) —
θ(j)− for j = 1, 2, respectively, and the total phase shifts of the N soliton pairs are
\[
\sum_{\ell=1}^{2} \sum_{j=1}^{N} \left( \theta^{(j)+} - \theta^{(j)-} \right)
\]
These quantities indicate that the N soliton pairs undergo elastic collisions.

Figure 3 depicts the two soliton pair solutions (2), comprising four individual bright solitons, two of which having equal amplitudes. Figure 4 displays the comparison between the exact two soliton pair solutions and their asymptotics, demonstrating a close coincidence between the exact solutions and their asymptotic counterparts.

Fig. 3 – (Colour online) The two soliton pair solitons in Eq. (2) with parameters
\[ N = 2, \epsilon = -1, p_1 = 2 + 2i, q_1 = 1 + i, p_2 = 3 + i, q_2 = 1 + 3i, \xi_{1,0} = 0, \eta_{1,0} = 0, \xi_{2,0} = 0, \eta_{2,0} = 0 \]
at times \( t = -2, 0, 2 \).

Fig. 4 – (Colour online) The comparison of the exact two soliton pair solitons in Eq. (2) (blue solid line) with their asymptotics (\( A^-, Q^- \)) outlined by Eq. (33) at time \( t = -5 \) (red dotted line) along \( y = 0 \) with the same parameters as in Fig. 3.

3. DERIVATIONS OF THE MULTIPLE BRIGHT SOLITON PAIR SOLUTIONS

This Section provides the derivations of the soliton solutions outlined as in Theorem 1 to the fully \( PT \)-symmetric nonlocal Davey–Stewartson I equation (1).
The multiple bright soliton pairs of the fully $PT$-symmetric nonlocal Davey–

Equation (1) is converted into the following bilinear equation

$$
(D_x^2 + D_y^2 - iD_t)g \cdot f = 0,
$$

$$
(D_x^2 - D_y^2)f \cdot f = -2\epsilon \hat{g}\hat{g},
$$

through the transformation in Eq. (2), where the operator $D$ is the Hirota’s bilinear
differential operator [39], and the function $f$ is subject to the following restriction:

$$
f(x, y, t) = f^*(-x, -y, t).
$$

To derive the bright soliton pair solutions in Theorem 1, we start with the fol-

lowing tau functions [39]

$$
\tau_0 = |\mathbf{M}|, \tau_1 = \begin{vmatrix} \mathbf{M} & \Phi^T \\ \Psi & 0 \end{vmatrix}, \tau_{-1} = \begin{vmatrix} \mathbf{M} & \Psi^T \\ -\Phi & 0 \end{vmatrix},
$$

where

$$
\mathbf{M} = (m_{i,j})_{2N \times 2N},
$$

$$
\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_{2N}), \Psi = (\Psi_1, \Psi_2, \cdots, \Psi_{2N}),
$$

$$
\bar{\Phi} = (\bar{\Phi}_1, \bar{\Phi}_2, \cdots, \bar{\Phi}_{2N}), \bar{\Psi} = (\bar{\Psi}_1, \bar{\Psi}_2, \cdots, \bar{\Psi}_{2N}),
$$

and

$$
m_{i,j} = \frac{e^{\xi_i + \bar{\xi}_j}}{p_i + \bar{p}_j} + \frac{e^{\eta_i + \bar{\eta}_j}}{q_i + \bar{q}_j},
$$

$$
\xi_i = e^{\xi_i}, \bar{\xi}_j = e^{\bar{\xi}_j}, \eta_i = e^{\eta_i}, \bar{\eta}_j = e^{\bar{\eta}_j},
$$

$$
\xi_i = p_i x_1 + p_i^2 x_2 + \xi_{i,0}, \bar{\xi}_j = p_j x_1 - p_j^2 x_2 + \bar{\xi}_{j,0},
$$

$$
\eta_i = q_i x_{-1} + q_i^2 x_{-2} + \eta_{i,0}, \bar{\eta}_j = q_j x_{-1} - q_j^2 x_{-2} + \bar{\eta}_{j,0}.
$$

The tau functions in Eq. (36) satisfy the following bilinear equations in the KP
hierarchy [39]

$$
(D_{x_1}^2 - D_{x_2})\tau_1 \cdot \tau_0 = 0,
$$

$$
(D_{x_1}^2 - D_{x_2})\tau_1 \cdot \tau_0 = 0,
$$

$$
(D_{x_1}D_{x_{-1}}^2 - 2)\tau_0 \cdot \tau_0 = -2\tau_1 \tau_{-1}.
$$

This set of bilinear equations can be reduced to the bilinear forms of the nonlocal
Davey–Stewartson I equation given by Eq. (34) under the variable transformation

$$
x_{-2} = \frac{i}{2} t, x_{-1} = -\frac{\epsilon}{2} (x - y), x_1 = \frac{1}{2} (x + y), x_2 = -\frac{i}{2} t,
$$

and the function transformation

$$
f = \tau_0, g = \tau_1, \hat{g} = \tau_{-1},
$$

if the tau functions $\tau_0, \tau_1, \tau_{-1}$ in Eq. (36) with the variable transformation in Eq. (40)
satisfy the symmetry condition
\[ \hat{\tau}_0(x, y, t) = \tau_0(x, y, t), \hat{\tau}_1(x, y, t) = -\tau_{-1}(x, y, t). \] (42)

Below, we proceed to realize this symmetry condition. For this purpose, we first set
the parameters \( p_i, p_i^*, q_i, q_i^*, \xi_i, \xi_i^*, \eta_i, \eta_i^* \) fulfills the complex conjugate condition
\[ p_i = p_i^*, q_i = q_i^*, \xi_i = \xi_i^*, \eta_i = \eta_i^* \] (43)
for \( i = 1, 2, \cdots, 2N \), and then further let the parameters
\[ p_j, p_{N+j}, q_j, q_{N+j}, \xi_j, \xi_{N+j}, \eta_j, \eta_{N+j}, j = 1, 2, \cdots, N \]
meet the restriction in Eq. (6). Therefore, the following symmetry will be realized
\[ \hat{\xi}_i(x, y, t) = \xi_{N+i}^*(x, y, t), \hat{\xi}_{N+i}(x, y, t) = \xi_i^*(x, y, t), \]
\[ \hat{\eta}_i(x, y, t) = \eta_{N+i}^*(x, y, t), \hat{\eta}_{N+i}(x, y, t) = \eta_i^*(x, y, t), \] (44)
and this implies
\[ \hat{m}_{N+i,N+j}(x, y, t) = -m_{i,j}(x, y, t), \hat{m}_{N+i,N+j}^*(x, y, t) = -m_{N+j,i}(x, y, t), \]
\[ \hat{\Phi}_i(x, y, t) = \Phi_{N+i}(x, y, t), \hat{\Phi}_i(x, y, t) = \Phi_{N+i}(x, y, t), \]
\[ \hat{\Psi}_i(x, y, t) = \Psi_{N+i}(x, y, t), \hat{\Psi}_i(x, y, t) = \Psi_{N+i}(x, y, t), \] (45)
Utilizing those symmetries, and performing simple row and column manipulations
for the determinants of \( \tau_0, \tau_1, \tau_{-1} \), we can quickly show that \( \tau_0, \tau_1, \tau_{-1} \) obey the
symmetry condition in Eq. (42). Accordingly, the tau functions \( \tau_0, \tau_1, \tau_{-1} \) in Eq. (36)
under parameters (6) and (43) reduce to the solutions of the bilinear equations ((34))
by taking the transformations in Eqs. (40) and (41). Finally, under the following scaling
\[ m_{i,j}e^{\eta_i + \eta_j} \rightarrow \hat{m}_{i,j}, \Phi_i e^{\eta_i} \rightarrow \Phi_i, \Psi_i e^{\eta_j} \rightarrow 1, \] (46)
then the tau functions \( \tau_0, \tau_1 \) in Eq. (36) become those given in Eq. (2). This com-
pletes the proof of Theorem 1.

4. CONCLUSION AND DISCUSSION

In this paper, we have studied the dynamics of multiple bright soliton pair in-
teractions in the fully \( \mathcal{PT} \)-symmetric nonlocal Davey–Stewartson I equation. First,
we have constructed the multiple bright soliton pair solutions on vanishing boundary
conditions to the fully \( \mathcal{PT} \)-symmetric nonlocal Davey–Stewartson I equation via the
bilinear KP-hierarchy reduction method. Then, the dynamics of the single soliton
pair solutions has been investigated by conducting the large time asymptotic analysis

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for them, and we found that the two solitons in each single soliton pair move parallel in the \((x, y)\) plane, but in two opposite directions. Finally, we have also performed the asymptotic analysis for the \(N\) individual soliton pair solutions as \(t \to \pm \infty\). Based on the asymptotic results, we found that i) these \(N\) individual soliton pairs only exhibit elastic collisions and ii) the two single solitons in each soliton pair have equal amplitudes.

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