STUDY OF THREE INTEGRABLE EXTENSIONS OF KADOMTSEV–PETVIASHVILI, BOUSSINESQ, AND KADOMTSEV–PETVIASHVILI–BOUSSINESQ EQUATIONS

ABDUL-MAJID WAZWAZ

Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
Email: *wazwaz@sxu.edu*

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Abstract. We study three (2 + 1)-dimensional extensions of Kadomtsev–Petviashvili (eKP) equation, Boussinesq (eBO) equation, and Kadomtsev–Petviashvili-Boussinesq (eKP-eBO) equation that appear in many physical settings in dissipative media. The Painlevé test is employed to confirm the integrability of each proposed model. We furnish dispersion relations, phase shifts, and multiple soliton solutions for each extended model. The bilinear form of each equation will be used to explore a class of lump solutions for these equations using distinct values of the parameters.

Key words: KP equation; Boussinesq equation; Painlevé analysis; soliton solutions; lump solutions.

1. INTRODUCTION

The standard integrable Kadomtsev–Petviashvili (KP) equation reads [1–4]:

$$(u_t + 6uu_x + u_{xxx})_x + \lambda u_{yy} = 0, \tag{1}$$

which admits weakly dispersive waves, with quadratic nonlinearity term $(uu_x)_x$ and a weak dispersion term u_{xxxx}. Equation (1) characterizes the phenomenon of small surface tension in comparison to the gravitational force in fluid dynamics and appears in many physical settings [5–13].

Moreover, the Boussinesq (BO) equation reads:

$$u_{tt} - u_{xxxx} - u_{xx} - 3(u^2)_{xx} = 0, \tag{2}$$

where $u(x,t)$ is a real-valued sufficiently often differentiable function that represents the height of the free surface of a fluid, and the subscripts denote the partial derivatives. The BO equation (2) models small-amplitude dispersive waves in shallow water, propagating in both the right and the left directions [1–4, 14–17]. The KP equation (1) and the BO equation (2) arise in a variety of physical applications such as iron sound waves in a plasma, the percolation of water in porous subsurface of a horizontal layer of material, coastal harbors, tsunami wave propagation, ocean beaches and others. Extensive research work has been employed to explore a huge number of exact solutions of the two well known equations.

Intensive research works have been invested on studying a variety of the KP equations (1) and the BO equations (2) that appear in (1+1) and higher dimensions. The standard KP equation (1) was derived to study the stability of the celebrated Korteweg-de Vries equation in two-dimensional media. The study of the KP equation, which describes a variety of scientific phenomena, such as, dust acoustic waves, fluid dynamics [18–28], weakly nonlinear quasi-unidirectional waves [29–34], and many others attracted a lot of research works. The KP equation is known to have families of exact solutions with distinct physical features.

Recently, an extended KP (eKP) equation was proposed as [1–4]

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + \lambda u_{tt} + \mu u_{ty} = 0, \tag{3}$$

where λu_{tt} and μu_{ty} are added to the standard KP equation (1), $u = u(x,y,t)$ is a differentiable function and λ and μ are non-zero constants. Another extension of the KP equation takes the form [35–45]:

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} - \frac{\alpha^2}{4} u_{tt} + \alpha u_{ty} + \beta u_{xt} = 0, \tag{4}$$

and was thoroughly examined in Refs. [1–4, 15–17].

A soliton solution is an analytic solution that is exponentially localized in all directions in space of x and y, and in time t [1,2,9,11,12]. Lump solutions, sometimes called rational function solutions, propagate with a higher propagating energy than the general solitons. Lumps are kinds of rational function solutions and have been thoroughly studied. They can be obtained *via* using the Hirota bilinear form. Using a dependent variable transformation gives a nonlinear equation in bilinear form, and hence, multiple soliton solutions and lump solutions can be readily obtained.

In this work, we plan to study three extensions of (2+1)-dimensional dispersive equations, given as

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{yy} + a u_{xx} + b u_{xy} = 0, \tag{5}$$

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{tt} + a u_{xx} + b u_{xy} = 0, \tag{6}$$

and

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{yy} + \gamma u_{tt} + a u_{xx} + b u_{xy} + c u_{yt} = 0, \tag{7}$$

which will be referred to as the extended KP (eKP) equation, the extended Boussinesq (eBO) equation, and the extended Kadomtsev–Petviashvili-Boussinesq (eKP-eBO) equation, respectively, where $u = u(x,y,t)$ is a differentiable function with respect to the spatial variables x and y, and the temporal variable t, and $\alpha, \beta, \gamma, a, b$, and c are arbitrary parameters. An identical model to equation (5) has been studied in Refs. [1–4, 7, 10, 13], but will be studied here with additional discussions for comparisons with the other two extensions. However, the second extension (6) is obtained from (5) only by replacing the term βu_{yy} by the term βu_{tt}. Moreover, it is obvious that equation (7) involves two terms added to Eq. (5), namely γu_{tt} and cu_{yt}.

This paper focuses primarily on the investigation of three forms of the KP-like equation. The overall structure of the paper is outlined as follows. Section 2 is dedicated to the investigation of the first extended model (5), the integrability through using Painlevé test, the multiple soliton solutions through using Hirota bilinear scheme, and the lump solutions. Sections 3 and 4 will follow the same analysis as Sec. 2. Lastly, the conclusions and the discussion of results are presented in Sec. 5.

2. EXTENDED (2+1)-DIMENSIONAL KP (EKP) EQUATION

We first examine an extended (2+1)-dimensional KP (eKP) equation:

$$u_{xt}+\alpha(u_{xxxx}+6u_xu_{xx})+\beta u_{yy}+au_{xx}+bu_{xy}=0, \tag{8}$$

where $u=u(x,y,t)$ is a differentiable function with respect to the spatial variables x and y, and the temporal variable t, and α,β,a, and b are arbitrary parameters. We plan to show that this equation is Painlevé integrable, hence we will derive multiple soliton solutions. Moreover, will be explored a class of lump solutions for distinct values of the involved parameters.

2.1. PAINLEVÉ ANALYSIS TO FIRST EXTENDED KP (EKP) EQUATION

The meaning of Painlevé integrability of any nonlinear partial differential equation is that the solution is single valued in the vicinity of a movable singularity manifold. Weiss, Tabor and Carnevale (WTC) [20] developed an algorithm (WTC method) to study the compatibility criteria for Painlevé integrability.

The Painlevé test consists of three main steps [1–4, 21, 25], which will be examined as follows:

(i) Leading order behavior and coefficients:

To obtain the leading order behavior and coefficients, we substitute the ansatz

$$u(x,y,t)=u_0\phi^{\alpha_1}, \tag{9}$$

into (8) to obtain

$$\alpha_1=-1, u_0=2\phi_x. \tag{10}$$

(ii) Resonant points:

We aim to determine the resonant points, which are those values of j at which it is possible to introduce arbitrary functions into the Laurent series

$$u(x,y,t)=\sum_{j=0}^{\infty}u_j\,\phi^{j+k}, \tag{11}$$

and it is single-valued in the neighborhood of singularity manifold ϕ. To achieve this aim, we insert

$$u(x,y,t) = u_0\phi^{-1} + u_j\phi^{j-1}, \tag{12}$$

into equation (8). Following the WTC analysis [20], and balancing the most dominant terms, we obtain the resonance points $-1,1,4,6$. This is due to the fourth-order of the linear structure of equation (8).

(iii) Verifying compatibility conditions

To verify the compatibility conditions, we follow the works [1–4, 20, 21, 25]. However, as usual, resonance at $k=-1$ corresponds to the arbitrariness of singular manifold $\psi(x,y,t)=0$. We also find explicit expressions for u_2, u_3, and u_5. However, we found that u_1, u_4, u_6 turn out to be arbitrary functions for all real values of the non-zero parameters. In view of this, the (2+1)-dimensional eKP extension (8) is Painlevé integrable.

2.2. MULTIPLE SOLITON SOLUTIONS

To derive the multiple soliton solutions of the extended eKPI equation (8), we substitute

$$u(x,y,t) = e^{k_i x + r_i y - c_i t}, \tag{13}$$

into the linear terms of (8) and we get the dispersion relations as

$$c_i = \frac{\alpha k_i^4 + \beta r_i^2 + a k_i^2 + b k_i r_i}{k_i}, i = 1,2,\cdots,N, \tag{14}$$

whic gives the phase variables by

$$\theta_i = k_i x + r_i y - \frac{\alpha k_i^4 + \beta r_i^2 + a k_i^2 + b k_i r_i}{k_i} t. \tag{15}$$

We next use the transformation

$$u(x,y,t) = 2(\ln f(x,y,t))_x, \tag{16}$$

into Eq. (8), where the auxiliary function $f(x,y,t)$, for the single soliton solution, is given as

$$f(x,y,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x + r_1 y - \frac{\alpha k_1^4 + \beta r_1^2 + a k_1^2 + b k_1 r_1}{k_1} t}. \tag{17}$$

This in turn leads to the single soliton solution

$$u(x,y,t) = \frac{2k_1 e^{k_1 x + r_1 y - \frac{\alpha k_1^4 + \beta r_1^2 + a k_1^2 + b k_1 r_1}{k_1} t}}{1 + e^{k_1 x + r_1 y - \frac{\alpha k_1^4 + \beta r_1^2 + a k_1^2 + b k_1 r_1}{k_1} t}}. \tag{18}$$

For the two-soliton solutions we use the auxiliary function as

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{19}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (19) into (8), and solving for the phase shift a_{12} we obtain

$$a_{12} = \frac{3\alpha k_1^2 k_2^2 (k_1 - k_2)^2 - \beta(k_1 r_2 - k_2 r_1)^2}{3\alpha k_1^2 k_2^2 (k_1 + k_2)^2 - \beta(k_1 r_2 - k_2 r_1)^2}, \tag{20}$$

which can be generalized as

$$a_{ij} = \frac{3\alpha k_i^2 k_j^2 (k_i - k_j)^2 - \beta(k_i r_j - k_j r_i)^2}{3\alpha k_i^2 k_j^2 (k_i + k_j)^2 - \beta(k_i r_j - k_j r_i)^2}, \; 1 \le i < j \le 3. \tag{21}$$

The result (21) shows that the phase shifts (21) depend on the parameters α and β as well as on the coefficients of the spatial parameters k_n, and $r_n, n = 1,2,3$. The two-soliton solutions are obtained by substituting (20) and (19) into (16).

For the three-soliton solutions, we apply the auxiliary function $f(x,y,t)$ as

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3} + b_{123} e^{\theta_1+\theta_2+\theta_3}. \tag{22}$$

Proceeding as before, we find

$$b_{123} = a_{12} a_{23} a_{13}. \tag{23}$$

The three-soliton solutions are obtained by substituting (22) into (16). This also shows that N-soliton solutions can be obtained for finite N, where $N \ge 1$.

2.3. LUMP SOLUTIONS

The lump solution is a kind of rational function solution localized in all spatial directions x and y, and propagate with higher propagating energy than the general solitons, and can be found when surface tension dominates the shallow water surface. In this Section we plan to derive a class of lump solutions for arbitrary values of the parameters. We first transform the (2+1)-dimensional eKP equation (8) into a bilinear equation in operators form given as

$$\left(D_x D_t + \alpha D_x^4 + \beta D_y^2 + a D_x^2 + b D_x D_y\right) f \cdot f = 0, \tag{24}$$

where D_t, D_x, and D_y are the Hirota's bilinear derivative operators. Equation (24) can be transformed to

$$\begin{aligned} &(f f_{xt} - f_x f_t) + \alpha(f f_{xxxx} - 4 f_{xxx} f_x + 3(f_{xx})^2) + \beta(f f_{yy} - f_y f_y) \\ &+ a(f f_{xx} - f_x f_x) + b(f f_{xy} - f_x f_y) = 0, \end{aligned} \tag{25}$$

obtained upon using

$$u(x,y,t) = 2(\ln f(x,y,t))_x. \tag{26}$$

To obtain the quadratic soliton solutions for the (2+1)-dimensional extended eKP equation (8), we set the following assumptions

$$\begin{array}{rcl} g &=& a_1x+a_2y+a_3t+a_4, \\ h &=& a_5x+a_6y+a_7t+a_8, \\ f &=& g^2+h^2+a_9, \end{array} \tag{27}$$

where $a_j, 1 \leq j \leq 9$ are real parameters that we will be formally determined. Substituting (27) in (25), we get a polynomial of the variables x, y, and t. To determine the parameters $a_j, 1 \leq j \leq 9$, we build up a system of equations of the coefficients of $t^2, xt, yt, zt, t, z^2, xz, yz, z, y^2$, xy, y, x^2, x, and the constant terms. By solving this system through using Maple, we obtain the following selected sets of constraining equations on the various parameters, noting that other sets can be obtained:

Case 1.

In this case we select, and hence obtain

$$\begin{array}{l} a_1=a_1, a_2=a_2, a_4=a_4, a_5=a_5, a_6=a_6, a_8=a_8, \\ a_3=\frac{\beta a_1(a_2^2-a_6^2)+(aa_1+ba_2)(a_1^2+a_5^2)+2\beta a_2a_5a_6}{a_1^2+a_5^2}, \\ a_7=-\frac{-\beta a_5(a_2^2-a_6^2)+(aa_5+ba_6)(a_1^2+a_5^2)+2\beta a_1a_2a_6}{a_1^2+a_5^2}, \\ a_9=-\frac{3\alpha(a_1^2+a_5^2)^3}{\beta(a_1a_6-a_2a_5)^2}, \alpha\beta<0, a_1a_6\neq a_2a_5. \end{array} \tag{28}$$

Notice that $\alpha\beta < 0$ so that $a_9 > 0$. The last results will guarantee a well-defined function $f(x,y,t)$, its positiveness, and the localization of $u(x,y,t)$ in all directions in the space, respectively. The obtained parameters (28) generate the class of positive quadratic function solutions upon substituting (28) in (27). This in turn will give a first class of lump solutions to the (2+1)-dimensional eKP equation (8) by using $u = 2(\ln f(x,y,t))_x$. Note that the obtained lump solutions $u(x,y,t) \to 0$ if and only if $g^2+h^2 \to \infty$.

For example, selecting

$$a_1=1, a_2=3, a_4=2, a_5=1, a_6=2, a_8=2, \alpha=-1, \beta=1, \tag{29}$$

gives

$$a_3=-a-3b-\frac{17}{2}, a_7=-a-2b-\frac{7}{2}, a_9=24. \tag{30}$$

This in turn gives the lump solution as

$$u(x,y,t)=\frac{2[4x+10y-4(a+\frac{5}{2}b+6)t+8]}{[x+3y+(-a-3b-\frac{17}{2})t+2]^2+[x+2y+(-a-2b-\frac{7}{2})t+2]^2+24}. \tag{31}$$

Figure 1 shows the lump solution (31) for $-5 \leq x,t \leq 7, \alpha=-1, \beta=1, a=b=1$.

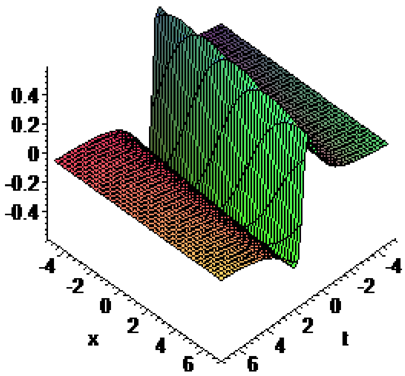

Fig. 1 – Profile of the lump solutions (31) for $-5 \leq x, t \leq 7, \alpha = -1, \beta = 1, a = b = 1$.

Case 2.

In this case we select:

$$\begin{aligned}
&a_1 = a_1, a_4 = a_4, a_8 = a_8,\\
&a_2 = \tfrac{\sqrt{a\beta}a_1a_8}{a_4\beta}, a_4\beta \neq 0,\\
&a_3 = -ba_2,\\
&a_5 = \tfrac{a_1a_8}{a_4}, a_4 \neq 0,\\
&a_6 = \tfrac{a_2a_4}{a_8}, a_8 \neq 0,\\
&a_7 = -ba_6,\\
&a_9 = -\tfrac{3a_1^2\alpha(a_4^2+a_8^2)}{aa_4^2}, a \neq 0.
\end{aligned} \tag{32}$$

Notice that $\alpha < 0$ and $aa_4 \neq 0$, so that $a_9 > 0$. The last results will guarantee a well-defined function $f(x,y,t)$, its positiveness, and the localization of $u(x,y,t)$ in all directions in the space, respectively.

For example, selecting

$$a_1 = 1, a_4 = 2, a_8 = 2, \alpha = -1, \beta = 1, \tag{33}$$

gives

$$a_2 = \sqrt{a}, a_3 = -b\sqrt{a}, a_5 = 1, a_6 = -\sqrt{a}, a_7 = b\sqrt{a}, a_9 = \frac{6}{a}, a > 0. \tag{34}$$

This in turn gives the lump solution as

$$u(x,y,t) = \frac{8(x+2)}{(x+\sqrt{a}\,y-\sqrt{a}\,bt+2)^2+(x-\sqrt{a}\,y+\sqrt{a}\,bt+2)^2+\frac{6}{a}}. \tag{35}$$

3. EXTENDED (2+1)-DIMENSIONAL BOUSSINESQ (EBO) EQUATION

We first examine an extended (2+1)-dimensional Boussinesq (eBO) equation

$$u_{xt}+\alpha(u_{xxxx}+6u_xu_{xx})+\beta u_{tt}+au_{xx}+bu_{xy}=0, \tag{36}$$

obtained by replacing u_{yy} in (8) by u_{tt}, where $u=u(x,y,t)$ is a differentiable function with respect to the spatial variables x and y, and the temporal variable t, and $\alpha,\beta,a,$ and b are arbitrary parameters. Following the analysis presented earlier, we will employ the Painlevé analysis to confirm its integrability. Based on this, we will explore multiple soliton solutions and a class of lump solutions as well.

3.1. PAINLEVÉ ANALYSIS FOR EXTENDED BOUSSINESQ (EBO) EQUATION

The meaning of Painlevé integrability of any nonlinear partial differential equation is that the solution is single valued in the vicinity of a movable singularity manifold. Weiss, Tabor, and Carnevale (WTC) [20] developed a powerful algorithm to study the compatibility criteria for Painlevé integrability. Following the three main steps of the Painlevé test as employed in the previous Section of the eKP case, we obtain identical results, hence we skip the details.

3.2. MULTIPLE SOLITON SOLUTIONS

To derive the multiple soliton solutions of the extended Boussinesq (eBO) equation (36), we substitute

$$u(x,y,t)=e^{k_ix+r_iy-c_it}, \tag{37}$$

into the linear terms of (36) and we get the dispersion relations as

$$c_i=-\frac{(-k_i+\sqrt{-4\alpha\beta k_i^4-4a\beta k_i^2-4b\beta k_ir_i+k_i^2})}{2\beta}, \beta\neq 0, i=1,2,\cdots,N, \tag{38}$$

which give the phase variables by

$$\theta_i=k_ix+r_iy+\frac{(-k_i+\sqrt{-4\alpha\beta k_i^4-4a\beta k_i^2-4b\beta k_ir_i+k_i^2})}{2\beta}t. \tag{39}$$

We next use the transformation

$$u(x,y,t)=2(\ln f(x,y,t))_x, \tag{40}$$

into Eq. (36), where the auxiliary function $f(x,y,t)$, for the single soliton solution, is given as

$$f(x,y,t) = 1+e^{\theta_1} = 1+e^{k_1x+r_1y+\frac{(-k_1+\sqrt{-4\alpha\beta k_1^4-4a\beta k_1^2-4b\beta k_1r_1+k_1^2})}{2\beta}t}. \tag{41}$$

Consequently, the single soliton solution

$$u(x,y,t) = \frac{2k_1e^{k_1x+r_1y+\frac{(-k_1+\sqrt{-4\alpha\beta k_1^4-4a\beta k_1^2-4b\beta k_1r_1+k_1^2})}{2\beta}t}}{1+e^{k_1x+r_1y+\frac{(-k_1+\sqrt{-4\alpha\beta k_1^4-4a\beta k_1^2-4b\beta k_1r_1+k_1^2})}{2\beta}t}} \tag{42}$$

follows immediately.

For the two-soliton solutions we use the auxiliary function as

$$f(x,y,t) \quad = \quad 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \tag{43}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (43) into (36), and solving for the phase shift a_{12} we obtain

$$a_{12} = \frac{4\alpha\beta k_1k_2(2k_1^2-3k_1k_2+2k_2^2)+2b\beta(k_1r_2+k_2r_1)+k_1k_2(4a\beta-1)+R_1R_2}{4\alpha\beta k_1k_2(2k_1^2+3k_1k_2+2k_2^2)+2b\beta(k_1r_2+k_2r_1)+k_1k_2(4a\beta-1)+R_1R_2}, \tag{44}$$

where

$$R_m = \sqrt{-4\alpha\beta k_m^4-4a\beta k_m^2-4b\beta k_mr_m+k_m^2} > 0,, m = 1,2. \tag{45}$$

which can be generalized as

$$a_{ij} = \frac{4\alpha\beta k_ik_j(2k_i^2-3k_ik_j+2k_j^2)+2b\beta(k_ir_j+k_2r_i)+k_ik_j(4a\beta-1)+R_iR_j}{4\alpha\beta k_ik_j(2k_i^2+3k_ik_j+2k_j^2)+2b\beta(k_ir_j+k_2r_j)+k_ik_j(4a\beta-1)+R_iR_j}, \quad 1 \le i < j \le 3. \tag{46}$$

The result (46) shows that the phase shifts (46) depend on the parametes α and β as well as on the coefficients of the spatial parameters k_n, and $r_n, n = 1,2,3$. The two-soliton solutions are obtained by substituting (44) and (43) into (40).

For the three-soliton solutions, we apply the auxiliary function $f(x,y,t)$ as

$$f(x,y,t) = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{47}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{48}$$

The three-soliton solutions are obtained by substituting (47) into (40). This also shows that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

3.3. LUMP SOLUTIONS

We plan in this Section to determine a class of lump solutions for arbitrary values of the parameters. We first transform the extended (2+1)-dimensional Boussinesq equation (36) into a bilinear equation in operators form given as

$$\left(D_xD_t+\alpha D_x^4+\beta D_t^2+aD_x^2+bD_xD_y\right)f\cdot f=0, \tag{49}$$

where D_t, D_x, and D_y are the Hirota's bilinear derivative operators. Equation (49) can be transformed to

$$\begin{array}{l}(ff_{xt}-f_xf_t)+\alpha(ff_{xxxx}-4f_{xxx}f_x+3(f_{xx})^2)+\beta(ff_{tt}-f_tf_t)\\ +a(ff_{xx}-f_xf_x)+b(ff_{xy}-f_xf_y)=0,\end{array} \tag{50}$$

obtained upon using

$$u(x,y,t)=2(\ln f(x,y,t))_x. \tag{51}$$

To obtain the quadratic soliton solutions for the (2+1)-dimensional extended eBO equation (36), we follow the analysis presented in the previous Section and set

$$\begin{array}{rcl}g&=&a_1x+a_2y+a_3t+a_4,\\ h&=&a_5x+a_6y+a_7t+a_8,\\ f&=&g^2+h^2+a_9,\end{array} \tag{52}$$

where $a_j, 1\leq j\leq 9$ are real parameters that we will be formally determined. Substituting (52) in (50), we get a polynomial of the variables x, y, and t. To determine the parameters $a_j, 1\leq j\leq 9$, we build up a system of equations of the coefficients of $t^2, xt, yt, zt, t, z^2, xz, yz, z, y^2$, xy, y, x^2, x, and the constant terms. By solving this system through using Maple, we obtain the following selected sets of constraining equations on the various parameters, noting that other sets can be obtained:

Case 1.

In this case we select

$$\begin{array}{l}a_1=a_1, a_3=a_3, a_4=a_4, a_5=a_5, a_7=a_7, a_8=a_8,\\ a_2=\frac{\beta a_1(a_3^2-a_7^2)+(aa_1+a_3)(a_1^2+a_5^2)+2\beta a_3a_5a_7}{b(a_1^2+a_5^2)}, b\neq 0,\\ a_6=-\frac{-\beta a_5(a_3^2-a_7^2)+(aa_5+ba_7)(a_1^2+a_5^2)+2\beta a_1a_3a_7}{b(a_1^2+a_5^2)},\\ a_9=-\frac{3\alpha(a_1^2+a_5^2)^3}{\beta(a_1a_6-a_2a_5)^2}, \alpha\beta<0, a_1a_6\neq a_2a_5.\end{array} \tag{53}$$

Notice that $\alpha\beta<0$ so that $a_9>0$. This is identical to the first case of the extended KP equation. The last results will guarantee a well-defined function $f(x,y,t)$, its positiveness, and the localization of $u(x,y,t)$ in all directions in the space, respectively. The obtained parameters (53) generate the class of positive quadratic function solutions upon substituting (53) in (52). This in turn will give a first class of lump solutions to the (2+1)-dimensional KP (36) equation (36) by using $u = 2(\ln f(x,y,t))_x$. Note that the obtained lump solutions $u(x,y,t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

For example, selecting

$$a_1 = 1, a_3 = 3, a_4 = 2, a_5 = 1, a_7 = 1, a_8 = 2, \alpha = -1, \beta = 1, \tag{54}$$

gives

$$a_2 = -\frac{a+10}{b}, a_6 = -\frac{a}{b}, a_9 = 6. \tag{55}$$

This in turn gives the lump solution as

$$u(x,y,t) = \frac{2[4x - \frac{4(a+5)}{b}y + 8t + 8]}{(x - \frac{a+10}{b}y + 3t + 2)^2 + (x - \frac{a}{b}y + t + 2)^2 + 6}. \tag{56}$$

Case 2.

In this case we find

$$\begin{aligned}
&a_4 = a_4, a_5 = a_5, a_6 = a_6, a_7 = a_7, a_8 = a_8,\\
&a_1 = 0,\\
&a_2 = -\tfrac{2\beta a_5 a_7(aa_5+ba_6)+\beta a_7^2(2\beta a_7+3a_5)+a_5^2(aa_5+ba_6+a_7)}{ba_5\sqrt{\beta(\beta a_7^2+a_5(aa_5+ba_6+a_7))}}, ba_5 \neq 0,\\
&a_3 = \tfrac{\sqrt{\beta(\beta a_7^2+a_5(aa_5+ba_6+a_7))}}{\beta}, \beta \neq 0,\\
&a_9 = -\tfrac{3\alpha a_5^4}{\beta a_7^2+a_5(aa_5+ba_6+a_7)}.
\end{aligned} \tag{57}$$

Notice that $\alpha < 0$ and $aa_4 \neq 0$, so that $a_9 > 0$. The last results will guarantee a well-defined function $f(x,y,t)$ and its positiveness.

For example, using

$$a_4 = 2, a_5 = 1, a_6 = 2, a_7 = 1, a_8 = 2, \alpha = -1, \beta = 1, a = 1, b = 3, \tag{58}$$

gives

$$a_2 = -3, a_3 = 3, a_9 = \frac{1}{3}. \tag{59}$$

This in turn gives the lump solution as

$$u(x,y,t) = \frac{4(x+2y+t+2)}{(-3y+3t+2)^2 + (x+2y+t+2)^2 + \frac{1}{3}}. \tag{60}$$

4. EXTENDED (2+1)-DIMENSIONAL EKP–EBO EQUATION

We finally study an extended (2+1)-dimensional eKP-eBO equation

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{yy} + \gamma u_{tt} + au_{xx} + bu_{xy} + cu_{yt} = 0, \tag{61}$$

where $u = u(x,y,t)$ is a differentiable function with respect to the spatial variables x and y, and the temporal variable t, and α,β,a,b, and c are arbitrary parameters. As stated earlier, this equation is obtained from the extended eKP equation (8) by adding the two linear terms γu_{tt} and cu_{yt}. We will follow the analysis presented earlier in the previous two Sections.

4.1. PAINLEVÉ ANALYSIS TO THE EXTENDED EKP–EBO EQUATION

Using Mathematica code, we obtain that Eq. (61) is integrable only if

$$\gamma = \frac{c^2}{4\beta}. \tag{62}$$

As a result, the extended eKP-eBO equation (61) can be set as

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{yy} + \frac{c^2}{4\beta} u_{tt} + a u_{xx} + b u_{xy} + c u_{yt} = 0, \beta \neq 0. \tag{63}$$

Weiss, Tabor, and Carnevale (WTC) [20] developed a powerful algorithm to study the compatibility criteria for Painlevé integrability. Following the three main steps of the Painlevé test as employed earlier, we obtain identical results as reported in the previous Section, hence we skip details.

4.2. MULTIPLE SOLITON SOLUTIONS

To derive the multiple soliton solutions of the extended eKP-eBO equation (63), we substitute

$$u(x,y,t) = e^{k_i x + r_i y - c_i t}, \tag{64}$$

into the linear terms of (63) and we get the dispersion relations as

$$c_i = \frac{2(\beta(k_i + c r_i) + \sqrt{-\alpha\beta c^2 k_i^4 + \beta k_i^2(\beta - a c^2) + \beta c k_i r_i (2\beta - bc)}\,)}{c^2}, \quad c \neq 0, i = 1,2,\cdots,N, \tag{65}$$

which give the phase variables by

$$\theta_i = k_i x + r_i y - \frac{2(\beta(k_i + c r_i) + \sqrt{-\alpha\beta c^2 k_i^4 + \beta k_i^2(\beta - a c^2) + \beta c k_i r_i (2\beta - bc)}\,)}{c^2} t. \tag{66}$$

We next use the transformation

$$u(x,y,t) = 2(\ln f(x,y,t))_x, \tag{67}$$

into Eq. (63), where the auxiliary function $f(x,y,t)$, for the single soliton solution, is given as

$$f(x,y,t) = 1+e^{\theta_1} = 1+e^{k_1x+r_1y-\frac{2(\beta(k_1+cr_1)+\sqrt{-\alpha\beta c^2k_1^4+\beta k_1^2(\beta-ac^2)+\beta ck_1r_1(2\beta-bc)})}{c^2}t}. \tag{68}$$

This in turn leads to the single soliton solution

$$u(x,y,t) = \frac{2k_1e^{k_1x+r_1y-\frac{2(\beta(k_1+cr_1)+\sqrt{-\alpha\beta c^2k_1^4+\beta k_1^2(\beta-ac^2)+\beta ck_1r_1(2\beta-bc)})}{c^2}t}}{1+e^{k_1x+r_1y-\frac{2(\beta(k_1+cr_1)+\sqrt{-\alpha\beta c^2k_1^4+\beta k_1^2(\beta-ac^2)+\beta ck_1r_1(2\beta-bc)})}{c^2}t}}. \tag{69}$$

For the two-soliton solutions we use the auxiliary function as

$$f(x,y,t) = 1+e^{\theta_1}+e^{\theta_2}+a_{12}e^{\theta_1+\theta_2}, \tag{70}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute (70) into (61), and solving for the phase shift a_{12} we obtain

$$a_{12} = \frac{2\beta k_1k_2(2\alpha c^2k_1^2-3\alpha c^2k_1k_2+2\alpha c^2k_2^2+ac^2-\beta)-\beta c(k_1r_2+k_2r_1)(-bc+2\beta)+2S_1S_2}{2\beta k_1k_2(2\alpha c^2k_1^2+3\alpha c^2k_1k_2+2\alpha c^2k_2^2+ac^2-\beta)-\beta c(k_1r_2+k_2r_1)(-bc+2\beta)+2S_1S_2}, \tag{71}$$

which can be generalized as

$$a_{ij} = \frac{2\beta k_ik_j(2\alpha c^2k_i^2-3\alpha c^2k_ik_j+2\alpha c^2k_j^2+ac^2-\beta)-\beta c(k_ir_j+k_jr_i)(-bc+2\beta)+2S_iS_j}{2\beta k_ik_j(2\alpha c^2k_i^2+3\alpha c^2k_ik_j+2\alpha c^2k_j^2+ac^2-\beta)-\beta c(k_ij_2+k_jr_i)(-bc+2\beta)+2S_iS_j}, \tag{72}$$

for $1 \leq i < j \leq 3$, and

$$S_i = \sqrt{-\beta k_i(\alpha c^2k_i^3+ac^2k_i+bc^2r_i-2\beta cr_i-\beta ki)} \tag{73}$$

The result (72) shows that the phase shifts depend on the parameters α, β,a,b,c, as well as on the coefficients of the spatial parameters k_n, and $r_n, n = 1,2,3$. The two-soliton solutions are obtained by substituting (71) and (70) into (67).

For the three-soliton solutions, we apply the auxiliary function $f(x,y,t)$ as

$$f(x,y,t) = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{74}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{75}$$

The three-soliton solutions are obtained by substituting (74) into (67). This also shows that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

4.3. LUMP SOLUTIONS

Lump solution was defined earlier in the previous two Sections. In this Section we plan to derive a class of lump solutions for arbitrary values of the parameters.

We first transform the (2+1)-dimensional KP equation (63) into a bilinear equation in operators form given as

$$\left(D_xD_t+\alpha D_x^4+\beta D_y^2+\frac{c^2}{4\beta}D_t^2+aD_x^2+bD_xD_y+cD_yD_t\right)f\cdot f=0, \qquad (76)$$

where D_t, D_x, and D_y are the Hirota's bilinear derivative operators. Equation (76) can be transformed to

$$\begin{array}{l}(ff_{xt}-f_xf_t)+\alpha(ff_{xxxx}-4f_{xxx}f_x+3(f_{xx})^2)+\beta(ff_{yy}-f_yf_y)\\ +\frac{c^2}{4\beta}(ff_{tt}-f_tf_t)+a(ff_{xx}-f_xf_x)+b(ff_{xy}-f_xf_y)+c(ff_{yt}-f_yf_t)=0,\end{array} \qquad (77)$$

obtained upon using

$$u(x,y,t)=2(\ln f(x,y,t))_x. \qquad (78)$$

To obtain the quadratic soliton solutions for the (2+1)-dimensional extended KP equation (63), we set the following assumptions

$$\begin{array}{rcl} g &=& a_1x+a_2y+a_3t+a_4,\\ h &=& a_5x+a_6y+a_7t+a_8,\\ f &=& g^2+h^2+a_9, \end{array} \qquad (79)$$

where $a_j, 1\le j\le 9$ are real parameters that we will be formally determined.

Substituting (79) in (77), we get a polynomial of the variables x, y, and t. To determine the parameters $a_j, 1\le j\le 9$, we build up a system of equations for the coefficients of $t^2, xt, yt, zt, t, z^2, xz, yz, z, y^2$, xy, y, x^2, x, and the constant terms. By solving this system through using Maple, we obtain the following selected sets of constraining equations on the various parameters, noting that other sets can be obtained as well.

Case 1.

In this case we select

$$\begin{array}{l} a_1=a_1, a_3=a_3, a_4=a_4, a_5=a_5, a_8=a_8,\\ a_2=-\frac{2a_5[-\frac{1}{2}a_5(ba_1+ca_3)+\frac{1}{2}\sqrt{4\beta a_1^3(aa_1+a_3)-ba_1^3(ba_1+2ca_3)}]+(ba_1+ca_3)(a_1^2+a_5^2)}{2\beta a_1^2},\\ \beta a_1\neq 0,\\ a_6=\frac{-\frac{1}{2}a_5(ba_1+ca_3)+\frac{1}{2}\sqrt{4\beta a_1^3(aa_1+a_3)-ba_1^3(ba_1+2ca_3)}}{\beta a_1},\\ a_7=\frac{a_3a_5}{a_1}, a_1\neq 0,\\ a_9=-\frac{12\alpha\beta a_1(a_1^2+a_5^2)}{4\beta aa_1-b^2a_1-2bca_3+4\beta a_3}, \alpha\beta<0. \end{array} \qquad (80)$$

Notice that $\alpha\beta<0$ so that $a_9>0$. The last results will guarantee a well-defined function $f(x,y,t)$, its positiveness, and the localization of $u(x,y,t)$ in all directions in the space, respectively. This in turn will give a first class of lump solutions to the (2+1)-dimensional KP (63) equation by using $u = 2(\ln f(x,y,t))_x$. Note that the obtained lump solutions $u(x,y,t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

For example, selecting

$$a_1 = 1, a_3 = 1, a_4 = 2, a_5 = 1, a_8 = 2, \alpha = -1, \beta = 1, a = 1, b = 1, c = 3, \quad (81)$$

gives

$$a_2 = -\frac{5}{2}, a_6 = -\frac{3}{2}, a_7 = 1, a_9 = 24. \quad (82)$$

This in turn gives the lump solution as

$$u(x,y,t) = \frac{2(4x - 8y + 4t + 8)}{(x - \frac{5}{2}y + t + 2)^2 + (x - \frac{3}{2}y + t + 2)^2 + 24}. \quad (83)$$

Figure 2 shows the lump solution (83) for $-20 \le x \le 20, t = y = 2, \alpha = -1, \beta = 1, \alpha = -1, \beta = 1, a = 1, b = 1, c = 3$.

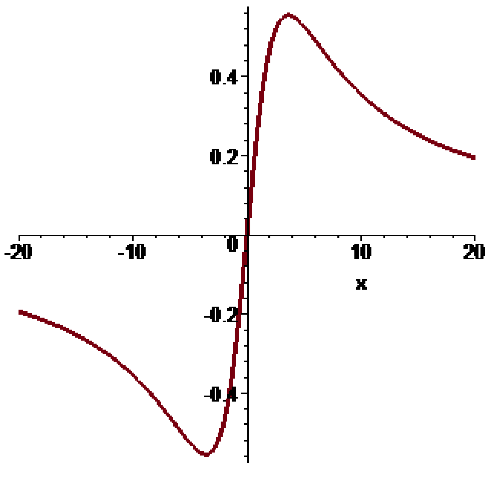

Fig. 2 – Profile of the lump solution (83) for
$-20 \le x \le 20, t = y = 2, \alpha = -1, \beta = 1, \alpha = -1, \beta = 1, a = 1, b = 1, c = 3$.

Case 2.

In this case we select

$$
\begin{aligned}
&a_1 = a_1, a_4 = a_4, a_8 = a_8,\\
&a_2 = \tfrac{\sqrt{a\beta}\, a_1 a_8}{a_4 \beta}, a_4\beta \neq 0, a\beta > 0,\\
&a_3 = -b a_2,\\
&a_5 = \tfrac{a_1 a_8}{a_4}, a_4 \neq 0,\\
&a_6 = \tfrac{a_2 a_4}{a_8}, a_8 \neq 0,\\
&a_7 = -b a_6,\\
&a_9 = -\tfrac{3a_1^2 \alpha (a_4^2 + a_8^2)}{a a_4^2}.
\end{aligned} \tag{84}
$$

Notice that $\alpha < 0$ and $aa_4 \neq 0$, so that $a_9 > 0$. The last results will guarantee a well-defined function $f(x,y,t)$, its positiveness, and the localization of $u(x,y,t)$ in all directions in the space, respectively. The obtained parameters (84) generate the class of positive quadratic function solutions upon substituting (80) in (79). This in turn will give a first class of lump solutions to the (2+1)-dimensional KP (63) equation by using $u = 2(\ln f(x,y,t))_x$.

For example, selecting

$$a_1 = 1, a_4 = 2, a_8 = 2, \alpha = -1, \beta = 1, \tag{85}$$

gives

$$a_2 = \sqrt{a}, a_3 = -b\sqrt{a}, a_5 = 1, a_6 = -\sqrt{a}, a_7 = b\sqrt{a}, a_9 = \frac{6}{a}, a > 0. \tag{86}$$

This in turn gives the lump solution as

$$u(x,y,t) = \frac{8(x+2)}{(x+\sqrt{a}\,y-\sqrt{a}\,bt+2)^2+(x-\sqrt{a}\,y+\sqrt{a}\,bt+2)^2+\frac{6}{a}}. \tag{87}$$

5. CONCLUSION

In this paper we studied three extensions of the KP, Boussinesq, and KP–Boussinesq equations, each of them in (2+1) dimensions. We employed the Painlevé analysis to confirm the integrability of the three extended models, and to confirm that the additional linear terms did not kill the integrability of each model. The Hirota's method was employed to exhibit multiple soliton solutions and lump solutions for each investigated model. We examined the parameters of each model to achieve the integrability and the desired solutions. The bilinear forms of the proposed models are presented by using Hirota's bilinear operator. Using a Maple symbolic computation, we were able to present a class of lump solutions for each examined equation.

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