INTRODUCING AND ANALYZING A NEW COMBINED VERSION OF THE UNSTABLE SCHRÖDINGER EQUATIONS WITH STRONG AND WEAK STABILITY EFFECTS

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In the literature, two types of unstable nonlinear Schrödinger equations have been independently developed and studied. Each was derived by incorporating either a self-effect term or a time-space dispersion term into the standard nonlinear Schrödinger equation. Both models describe the time evolution of disturbances in unstable media. The primary contribution of this work is the combination of these two types into a single, new unstable version of the nonlinear Schrödinger equation. This new model is analyzed using two effective methods: the rational sine-cosine and the rational sinh-cosh functions. Additionally, a comparison test of the embedded unstable terms is conducted to assess their respective impacts on the stability of the Schrödinger model. Finally, graphical analyses, including 2D and 3D plots, are performed to validate the study's findings.

Key words: Revisited unstable nonlinear Schrödinger equation, Modified rational sine-cosine method, Modified rational sinh-cosh method.

1. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) describes the evolution of slowly varying wave packets and generic small amplitude in a nonlinear medium [1]:

$$i\theta_t + \theta_{xx} + \alpha|\theta|^2\theta = 0, \tag{1}$$

where $\alpha \neq 0$ is a real constant and $\theta = \theta(x,t)$ is a complex-valued function. This equation has been derived in many physical settings such as plasma [2], deep water [3], and nonlinear optics [4]. By adding the term $\beta\theta$ to (1) it results in the first unstable nonlinear Schrödinger equation (uNLSE-I) that reads as:

$$i\theta_t + \theta_{xx} + \alpha|\theta|^2\theta + \beta\theta = 0. \tag{2}$$

On the other side, by adding the term $\gamma\theta_{xt}$ to (1) it results in the second unstable nonlinear Schrödinger equation (uNLSE-II) that reads as:

$$i\theta_t + \theta_{xx} + \alpha|\theta|^2\theta + \gamma\theta_{xt} = 0. \tag{3}$$

The uNLSE-I describes the time evolution of disturbances in unstable media. It has been studied in many occasions, for example, the (G'/G)-expansion method, the extended form of simple equation method, and various solutions were discovered for this type of unstable nonlinear Schrödinger equation [5, 6]. The uNLSE-II was proposed and investigated in [7–9], where the modified extended mapping method, the variational principle method, and the modified exponential rational function method were implemented and both elliptic as well as periodic solitary wave solutions were reported.

In the present work, we aim to combine both uNLSE-I and uNLSE-II as a new modified unstable nonlinear Schrödinger equation (muNLSE). The mathematical formulation of the muNLSE will read as

$$i\theta_t + \theta_{xx} + \alpha|\theta|^2\theta + \beta\theta + \gamma\theta_{xt} = 0. \tag{4}$$

Two objectives will be achieved for the proposed new model (4): discovering new rational form solutions in terms of sine-cosine and sinh-cosh functions by utilizing the proposed methodologies, and examining the effectiveness of the two unstable terms on the propagation in the framework of the muNLSE.

In the literature, researchers have proposed several powerful methods to discover exact solutions including solitons and others types related to nonlinear evolution equations and systems. Such methods are: the Backlund transform [10–12], the nonlinear transform method [13, 14], the Hirota direct method [15–18], the sine-cosine function method [19–21], tanh-coth expansion method [22, 23], Kudryashov-expansion [24–26], (G'/G)-expansion [27, 28], exp-function method [29, 30], rational sine-cosine method [31–33] and many others [34–43].

The paper is organized as follows: In Sec. 2, we establish the necessary conditions ensuring the existence of solitary wave solutions for the muNLSE. Section 3 employs the modified rational sine-cosine functions method to solve the proposed model. Also, additional solutions will be derived using the modified rational sinh-cosh functions method in Sec. 4. The dynamical behaviors of muNLSE will be investigated in Sec. 5. Finally, the findings of the paper are summarized in Sec. 6.

2. NECESSARY CONDITIONS FOR SOLITARY WAVE SOLUTIONS TO EXIST

The function $\theta = \theta(x,t)$ in (4) is a complex-valued function, which can be expressed as [44]:

$$\theta(x,t) = e^{i\zeta}\psi(z), \tag{5}$$

where $\zeta = \lambda(x + wt)$ and $z = x - ct$. Substituting (5) into (4) yields the following ordinary differential equation (ODE):

$$\begin{aligned}(c\gamma - 1)\psi''(z) \quad &+ \quad i(\gamma c\lambda + c - \lambda(\gamma w + 2))\psi'(z) + (\lambda(\lambda + \gamma\lambda w + w) - \beta)\psi(z) \\ &- \quad \alpha\psi^3(z) = 0 \end{aligned} \tag{6}$$

Since the coefficient of the second term of (6) is complex, it must be identical to zero. Consequently, under the following necessary condition

$$i(\gamma c\lambda + c - \lambda(\gamma w + 2)) = 0, \tag{7}$$

we only concern with the solution of the following ODE:

$$(c\gamma - 1)\psi''(z) + (\lambda(\lambda + \gamma\lambda w + w) - \beta)\psi(z) - \alpha\psi^3(z) = 0 \tag{8}$$

Next, we solve (8) using the rational sine-cosine and rational sinh-cosh methods.

3. THE RATIONAL SINE-COSINE APPROACH

This approach assumes that the solution of (8) is of the following form [45–47]:

$$\psi(z) = \frac{A + B\sin(\mu z)}{F + G\cos(\mu z)} : \quad \cos(\mu z) \neq -\frac{F}{G}. \tag{9}$$

To determine the values of the scalars A, B, F, G, μ, λ, w and c, we plug (9) into (8) and proceed to eliminate the coefficients of the resulting basis functions $\{\sin^i(\mu z)\cos^j(\mu z)\}_{i=0,1,\ j=0,1}$, $\sin^2(\mu z)$, and $\sin^3(\mu z)$. This leads to a set of six nonlinear algebraic equations:

$$\begin{aligned}
0 \quad &= \quad \frac{1}{2}BFG\left(2\beta - c\gamma\mu^2 - 2\lambda^2 + \mu^2 - 2\lambda w(\gamma\lambda + 1)\right), \\
0 \quad &= \quad AFG\left(2\beta - c\gamma\mu^2 - 2\lambda^2 + \mu^2 - 2\lambda w(\gamma\lambda + 1)\right), \\
0 \quad &= \quad B\left(\alpha B^2 + G^2(\lambda(\lambda + \gamma\lambda w + w) - \beta)\right), \\
0 \quad &= \quad A\left(3\alpha B^2 + G^2\left(-\beta - c\gamma\mu^2 + \lambda^2 + \mu^2 + \lambda w(\gamma\lambda + 1)\right)\right), \\
0 \quad &= \quad B\left(3\alpha A^2 + F^2\left(\beta + c\gamma\mu^2 - \lambda^2 - \mu^2 - \lambda w(\gamma\lambda + 1)\right)\right) \\
&+ \quad BG^2\left(\beta - 2c\gamma\mu^2 - \lambda^2 + 2\mu^2 - \lambda w(\gamma\lambda + 1)\right), \\
0 \quad &= \quad A\left(\alpha A^2 + G^2\left(\beta - c\gamma\mu^2 - \lambda^2 + \mu^2 - \lambda w(\gamma\lambda + 1)\right)\right) \\
&+ \quad AF^2(\beta - \lambda(\lambda + \gamma\lambda w + w)).
\end{aligned} \tag{10}$$

Then, we solve the system that includes both (10) and the constraint relation (7). As a result, we get the following families of solutions illustrated as below:

Family 1.1:

$$
\begin{aligned}
A &= \mp\frac{\sqrt{\frac{\mu^2(\beta\gamma^2-1)(G^2-F^2)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}}{\sqrt{\alpha}} \quad , \quad B=\mp\frac{G\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}}\\
c &= \frac{2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2} \quad , \quad w=\frac{2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2}{\lambda\left(\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2\right)}.
\end{aligned}
\tag{11}
$$

Therefore, the first solution to muNLSE is:

$$
\begin{aligned}
\theta_1(x,t) &= \frac{\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}}\, e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}+\lambda x\right)}\\
&\times \frac{\left(\mp\sqrt{\frac{\mu^2(G^2-F^2)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}\mp\frac{G\mu\sin\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}\right)\right)}{\sqrt{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}\right)}{\left(F+G\cos\left(\mu\left(x-\frac{t(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)\right)},
\end{aligned}
\tag{12}
$$

where $\alpha>0$ and $|G|>|F|$.

Family 1.2:

$$
\begin{aligned}
A &= F=0, \qquad B=\mp\frac{G\mu\sqrt{2\beta\gamma^2-2}}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1}},\\
c &= \frac{\beta\gamma+\gamma\lambda^2+2\gamma\mu^2+2\lambda}{\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1}, \quad w=\frac{\beta\gamma\lambda+\beta-\gamma\lambda^3-2\gamma\lambda\mu^2-\lambda^2+2\mu^2}{\lambda\left(\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1\right)}.
\end{aligned}
\tag{13}
$$

Consequently, the second solution to muNLSE is:

$$
\begin{aligned}
\theta_2(x,t) &= \mp\, e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3-2\gamma\lambda\mu^2-\lambda^2+2\mu^2\right)}{\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1}+\lambda x\right)}\\
&\times \frac{\mu\sqrt{2\beta\gamma^2-2}\tan\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2+2\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2+2\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1}}.
\end{aligned}
\tag{14}
$$

Family 1.3:

$$
\begin{aligned}
A &= \mp B = \mp \frac{G\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}, \qquad F=0,\\
c &= \frac{2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}, \qquad w=\frac{2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2}{\lambda(\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2)}.
\end{aligned} \tag{15}
$$

As a result, the third solution to muNLSE is:

$$
\begin{aligned}
\theta_3(x,t) &= \mp \frac{\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}\, e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}+\lambda x\right)}\\
&\times\ \sec\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)\\
&\times\ \left(1\mp\sin\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)\right).
\end{aligned} \tag{16}
$$

Family 1.4:

$$
\begin{aligned}
A &\quad \mp \frac{G\mu\sqrt{2\beta\gamma^2-2}}{\sqrt{\alpha}\sqrt{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}}, \qquad B=F=0,\\
c &= \frac{\beta\gamma+\gamma\lambda^2-\gamma\mu^2+2\lambda}{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}, \qquad w=\frac{\beta\gamma\lambda+\beta-\gamma\lambda^3+\gamma\lambda\mu^2-\lambda^2-\mu^2}{\lambda(\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1)}.
\end{aligned} \tag{17}
$$

Thus, the fourth solution to muNLSE is:

$$
\begin{aligned}
\theta_4(x,t) &= \mp\, e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3+\gamma\lambda\mu^2-\lambda^2-\mu^2\right)}{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}+\lambda x\right)}\\
&\times\ \frac{\mu\sqrt{2\beta\gamma^2-2}\sec\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2-\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}}.
\end{aligned} \tag{18}
$$

Family 1.5:

$$
\begin{aligned}
A &= 0, \qquad B=\mp\frac{G\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}, \qquad F=-G,\\
c &= \frac{2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}, \quad w=\frac{2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2}{\lambda(\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2)}.
\end{aligned} \tag{19}
$$

The fifth solution to muNLSE is:

$$\begin{aligned}\theta_5(x,t) &= \mp e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\ &\times \frac{\mu\sqrt{\beta\gamma^2-1}\,\cot\left(\frac{1}{2}\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}}. \end{aligned} \tag{20}$$

Family 1.6:

$$\begin{aligned} A &= 0, \qquad B=\mp\frac{G\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}}, \qquad F=G, \\ c &= \frac{2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}, \quad w=\frac{2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2}{\lambda\left(\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2\right)}. \end{aligned} \tag{21}$$

The sixth solution to muNLSE is:

$$\begin{aligned}\theta_6(x,t) &= \mp e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\ &\times \frac{\mu\sqrt{\beta\gamma^2-1}\,\tan\left(\frac{1}{2}\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}}. \end{aligned} \tag{22}$$

It is important to note that the solutions obtained using the sine-cosine approach are valid for any choice of the two parameters μ and λ. Specifically, selecting a particular value for λ yields an interesting result as stated in the following corollary.

Corollary 3.1. For each of the above families, Family 1.1 - Family 1.6, choosing $\lambda=-\frac{1}{\gamma}$ results in $w=-\frac{2}{\gamma}$.

REMARK 1. By swapping the sine and cosine functions in (9) for solving the muNLSE, *i.e.*, using $\psi(z)=\frac{A+B\cos(\mu z)}{F+G\sin(\mu z)}$, we obtain the same system as described in (10). Consequently, four additional new solutions can be extracted. These solutions are labeled as θ_7-θ_{10} and are defined as follows:

$$\theta_7(x,t) = e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}+\lambda x\right)}$$
$$\times \frac{\left(\mp\sqrt{\frac{\mu^2(\beta\gamma^2-1)(G^2-F^2)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}\mp\frac{G\mu\sqrt{\beta\gamma^2-1}\cos\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}}\right)}{\sqrt{\alpha}\left(F+G\sin\left(\mu\left(x-\frac{t(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda)}{\gamma^2(2\lambda^2+\mu^2)+4\gamma\lambda+2}\right)\right)\right)}.$$

$$\theta_8(x,t) = \mp e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3-2\gamma\lambda\mu^2-\lambda^2+2\mu^2\right)}{\gamma^2(\lambda^2+2\mu^2)+2\gamma\lambda+1}+\lambda x\right)}$$
$$\times \frac{\mu\sqrt{2\beta\gamma^2-2}\cot\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2+2\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2+2\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2+2\mu^2\right)+2\gamma\lambda+1}}. \tag{23}$$

$$\theta_9(x,t) = \mp\frac{\mu\sqrt{\beta\gamma^2-1}}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}}\, e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3-\gamma\lambda\mu^2-2\lambda^2+\mu^2\right)}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}+\lambda x\right)}$$
$$\times \csc\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}\right)\right)$$
$$\times \left(1\mp\cos\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2+\gamma\mu^2+4\lambda\right)}{\gamma^2\left(2\lambda^2+\mu^2\right)+4\gamma\lambda+2}\right)\right)\right).$$

$$\theta_{10}(x,t) = \mp e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3+\gamma\lambda\mu^2-\lambda^2-\mu^2\right)}{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}+\lambda x\right)}$$
$$\times \frac{\mu\sqrt{2\beta\gamma^2-2}\csc\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2-\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2-\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2-\mu^2\right)+2\gamma\lambda+1}}. \tag{24}$$

4. THE RATIONAL SINH-COSH APPROACH

Having successfully derived solutions to the muNLSE using rational sine-cosine functions, we will now explore the benefits of employing rational form solutions that incorporate the hyperbolic functions sinh and cosh. Specifically, we consider the solution of (8) in the form of:

$$\psi(z) = \frac{A+B\sinh(\mu z)}{F+G\cosh(\mu z)}: \ \cosh(\mu z)\neq -\frac{F}{G},$$
$$\text{or}$$
$$= \frac{A+B\cosh(\mu z)}{F+G\sinh(\mu z)}: \ \sinh(\mu z)\neq -\frac{F}{G}. \tag{25}$$

To determine the unknown A, B, F, G, μ, λ, w, and c, we follow the same steps outlined in the previous Section, but now using bases in terms of $\sinh(\mu z)$ and $\cosh(\mu z)$. For brevity, we omit the resulting algebraic system and proceed directly to present the attained new solutions as defined by:

$$\begin{aligned}
\theta_{11}(x,t) &= \sqrt{\beta\gamma^2-1}\, e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3+\gamma\lambda\mu^2-2\lambda^2-\mu^2\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\
&\times \frac{\left(\mp\frac{\sqrt{\frac{\mu^2(F^2-G^2)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}}}{\sqrt{\alpha}} \mp \frac{G\mu\sinh\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}}\right)}{F+G\cosh\left(\mu\left(x-\frac{t(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}. \\
\theta_{12}(x,t) &= \sqrt{\beta\gamma^2-1}\, e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3+\gamma\lambda\mu^2-2\lambda^2-\mu^2\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\
&\times \frac{\left(\mp\frac{\sqrt{\frac{\mu^2(F^2+G^2)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}}}{\sqrt{\alpha}} \mp \frac{G\mu\cosh\left(\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}}\right)}{F+G\sinh\left(\mu\left(x-\frac{t(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}.
\end{aligned} \tag{26}$$

$$\begin{aligned}
\theta_{13}(x,t) &= \mp\, e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3-\gamma\lambda\mu^2-\lambda^2+\mu^2\right)}{\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda+1}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{2-2\beta\gamma^2}\ \operatorname{sech}\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2+\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2+\mu^2\right)+2\gamma\lambda+1}}. \\
\theta_{14}(x,t) &= \mp\, e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3-\gamma\lambda\mu^2-\lambda^2+\mu^2\right)}{\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda+1}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{2\beta\gamma^2-2}\ \operatorname{csch}\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2+\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2+\mu^2\right)+2\gamma\lambda+1}}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\theta_{15}(x,t) &= \mp e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3+2\gamma\lambda\mu^2-\lambda^2-2\mu^2\right)}{\gamma^2(\lambda^2-2\mu^2)+2\gamma\lambda+1}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{2\beta\gamma^2-2}\,\tanh\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2-2\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2-2\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2-2\mu^2\right)+2\gamma\lambda+1}}. \\
\theta_{16}(x,t) &= \mp e^{i\left(\frac{t\left(\beta\gamma\lambda+\beta-\gamma\lambda^3+2\gamma\lambda\mu^2-\lambda^2-2\mu^2\right)}{\gamma^2(\lambda^2-2\mu^2)+2\gamma\lambda+1}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{2\beta\gamma^2-2}\,\coth\left(\mu\left(x-\frac{t\left(\beta\gamma+\gamma\lambda^2-2\gamma\mu^2+2\lambda\right)}{\gamma^2(\lambda^2-2\mu^2)+2\gamma\lambda+1}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(\lambda^2-2\mu^2\right)+2\gamma\lambda+1}}. \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\theta_{17}(x,t) &= \mp e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3+\gamma\lambda\mu^2-2\lambda^2-\mu^2\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{\beta\gamma^2-1}\coth\left(\frac{1}{2}\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2-\mu^2\right)+4\gamma\lambda+2}}. \\
\theta_{18}(x,t) &= \mp e^{i\left(\frac{t\left(2\beta(\gamma\lambda+1)-2\gamma\lambda^3+\gamma\lambda\mu^2-2\lambda^2-\mu^2\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}+\lambda x\right)} \\
&\times \frac{\mu\sqrt{\beta\gamma^2-1}\tanh\left(\frac{1}{2}\mu\left(x-\frac{t\left(2\beta\gamma+2\gamma\lambda^2-\gamma\mu^2+4\lambda\right)}{\gamma^2(2\lambda^2-\mu^2)+4\gamma\lambda+2}\right)\right)}{\sqrt{\alpha}\sqrt{\gamma^2\left(2\lambda^2-\mu^2\right)+4\gamma\lambda+2}}. \qquad (29)
\end{aligned}$$

5. THE DYNAMICS OF MUNLSE *VIA* GRAPHICAL ANALYSIS

The aim of this Section is twofold. First, we present the different types of propagation within the reported solutions in this work. Second, we examine the impact of the two distinct sources, $\beta\theta(x,t)$ and $\gamma\theta_{xt}(x,t)$, on the dynamic behavior of the NLSE within the newly proposed muNLSE model. This analysis will be conducted by visualizing the propagation in the framework of the muNLSE for various selections of the parameters β and γ.

The squared norm of a complex function holds substantial significance in physics, such as determining the probability density of locating a particle at position x and time t in quantum mechanics, or representing the intensity of an electromagnetic wave. Here, we identify the physical structures of the squared norm of the solutions obtained for the muNLSE, summarized as follows:

- The evolutions of the functions θ_1 through θ_{10} are moving convex-periodic waves, see Fig. 1 that depicts the plots of θ_3.

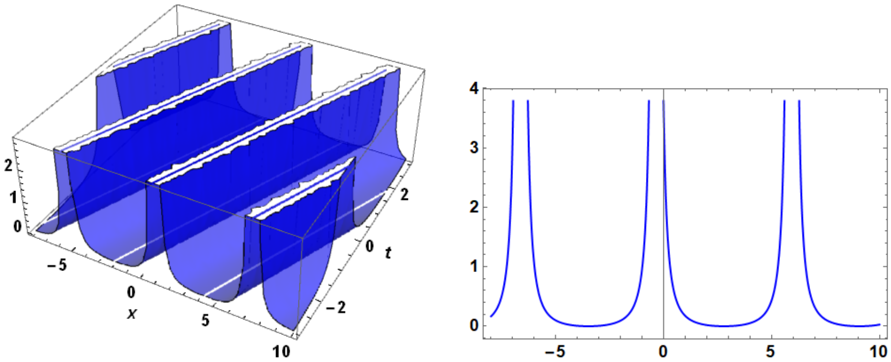

Fig. 1 – The motion of convex-periodic wave is illustrated for $\theta_3(x,t)$ with the parameters set as $\alpha = \gamma = \mu = \lambda = 1$ and $\beta = 2$.

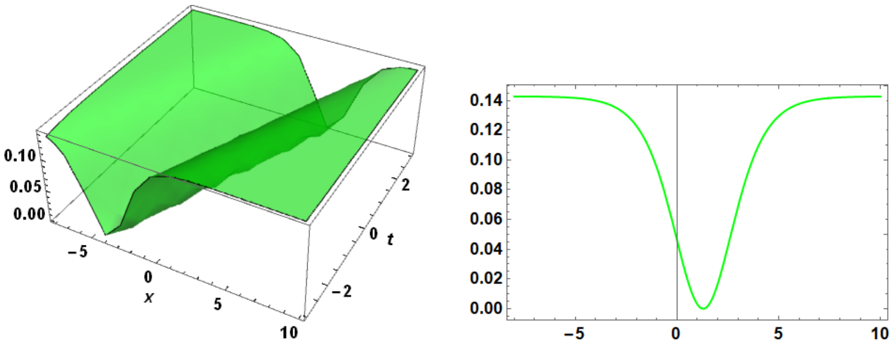

Fig. 2 – The motion of anti-bell-shaped wave is illustrated for $\theta_{18}(x,t)$ with the parameters set as $\alpha = \gamma = \mu = \lambda = 1$ and $\beta = 2$.

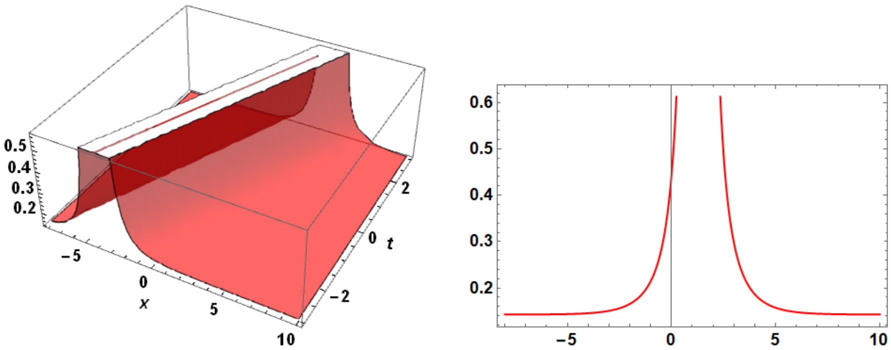

Fig. 3 – The motion of cusp-like singular wave is illustrated for $\theta_{17}(x,t)$ with the parameters set as $\alpha = \gamma = \mu = \lambda = 1$ and $\beta = 2$.

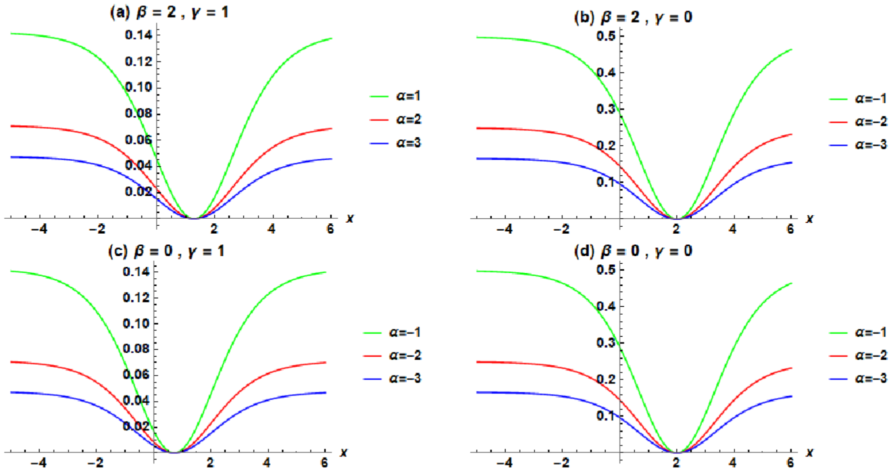

Fig. 4 – The impact of β and γ on the propagation of $\theta_9(x,t)$. Where $\gamma = \mu = \lambda = 1$.

- The functions θ_{15} and θ_{18} describe the motion of anti-bell-shaped wave, see Fig. 2 that depicts the plots of θ_{18}.
- Moving as cusp-like singular soliton is the nature of each of the functions θ_{11}-θ_{14}, θ_{16}-θ_{17}, see Fig. 3 that depicts the plots of θ_{17}.

As previously mentioned in this paper, the new model of the unstable NLSE incorporates two different sources compared to the standard NLSE. Here, we present our findings on the impact of these sources on the propagation of the traveling wave solutions obtained for the muNLSE. Our observations are as follows:

- In the absence of β, the parameter γ influences the propagation, as shown in Fig. 4 (c) and (d).
- In the absence of γ, the parameter β does not influence the propagation, as shown in Fig. 4 (b) and (d).
- In the presence of γ, the parameter β does not influence the propagation, as shown in Fig. 4 (a) and (c).

We may conclude that the NLSE is strongly unstable with the presence of γ only, and weakly unstable in the presence of β only.

6. CONCLUSION

In this paper, we introduced a new version of the unstable nonlinear Schrödinger equation, abbreviated as muNLSE. This model combines two existing unstable versions of the classical nonlinear Schrödinger equation. We identified three different physical structures for the muNLSE, which are classified as convex-periodic motion, anti-bell-shaped waves, and cusp-like singular waves. Graphical analyses were conducted, indicating that uNLSE-II exhibits strong instability, while uNLSE-I is weakly unstable.

For future research, we propose applying the Hirota bilinear approach to explore the existence of lump, breather, and rogue solutions [48–51], which could provide valuable insights into the broader dynamics of the muNLSE.

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