THE INTEGRABLE HIERARCHY AND THE NONLINEAR RIEMANN-HILBERT PROBLEM ASSOCIATED WITH ONE TYPICAL EINSTEIN-WEYL PHYSICO-GEOMETRIC DISPERSIONLESS SYSTEM

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Compiled April 27, 2025

Abstract. From a specific series of exchange conditions for a one-parameter Hamiltonian vector field, we establish an integrable hierarchy using Lax pairs derived from the dispersionless partial differential equation. An exterior differential form of the integrable hierarchy is introduced, further confirming the existence of the τ function. Subsequently, we present the twistor structure of the hierarchy. By constructing the nonlinear Riemann-Hilbert problem for the equation, the structure of the solution to the equation is better understood.

Key words: integrable hierarchy, Hamiltonian vector field, τ function, twistor structure, nonlinear Riemann-Hilbert problem.

1. INTRODUCTION

Einstein-Weyl geometry was first introduced by Weyl in the early 20th century and has been extensively studied [1, 2]. The relations of dispersionless integrable systems to Einstein-Weyl geometry have been discussed in [3–6], and there are several classes of partial differential equations (PDEs) whose integrability can be seen in their formal linearized geometries [7]. One class of integrable models possessing the central quadric ansatz is classified by the method of hydrodynamic reductions [8], leading to five regular forms. Then, applying the central quadric transformation can be obtained all Painlevé equations $P_I - P_{VI}$. These five regular forms appear in different forms in the classification of multidimensional integrable systems, including the well-known BF (Boyer-Finley) equation

$$(e^u)_{tt} + u_{xx} + u_{yy} = 0,$$

and the dKP (dispersionless Kadomtsev-Petviashvili) equation

$$u_{xt} - (uu_x)_x - u_{yy} = 0.$$

In this paper, we will focus on the equation related to physico-geometric systems as follows

$$u_{xt} + \frac{1}{2}u_{yy} + \frac{1}{2}\left(u^2\right)_{xy} = 0. \tag{1}$$

It is the representative case in the classification of infinite hydrodynamic chains satisfying the Egorov property. The concept of Egorov hydrodynamic chains has been introduced in [9, 10], which establishes their connection to the integrable (2+1)-dimensional equations of hydrodynamic type. The results of the classification problem for integrable (2+1)-dimensional hydrodynamic type systems are presented. In fact, the conformal structures corresponding to this integrable equation satisfy the Einstein-Weyl property. This construction uses the gauge group SDiff (Σ^2) modelled on Riccati spaces and provides an Einstein-Weyl structure from the solutions of gauge field equations [11]. Calderbank [11] pointed out that, based on generalized Nahm equations with the gauge group SDiff (Σ^2), can be obtained the generalized Nahm equations corresponding to equation (1) by twistor theoretical approach.

The study of the integrable systems originated in the 19th century in the field of mathematical physics, and it was an essential investigation of nonlinear evolution equations. These integrable systems have special nonlinear evolution equations in which the shape and behaviour of the waves remain constant throughout their evolution [12–23]. In the 1970s, it was observed that some nonlinear equations possessed more symmetries than the traditional Lie group symmetries [24]. Later, Mikhailov, Shabat, and Yamilov used a bilinear approach based on the Lax pairs, after which an infinite number of conservation laws and symmetries could be extracted from the nonlinear problem by introducing suitable variables [25]. In the 1990s, the Lax equation and Hamiltonian structures of the dispersionless integrable systems attracted increasing attention [26]. These works provide a foundation for the study of the properties of dispersionless equations, especially their infinite symmetries and the corresponding hierarchies. Actually, the integrable hierarchies with Hamiltonian vector field forms are more abundantly studied, for instance, the dKP hierarchies, SDiff(2) Toda hierarchies [27–29], and the dDS (dispersionless Davey-Stewartson) hierarchies [30].

Sato theory is widely recognized as one of the key theories in the field of integrable systems. Specifically, the τ function theory, which is the core of Sato theory, is essential to many related subjects and facilitates the analysis of PDEs. As an effective tool for studying nonlinear PDEs, the τ function theory covers a variety of topics including soliton theory, inverse scattering methods, Hamiltonian methods, etc. [31–33]. In addition to its applications in integrable systems, the τ function theory has significant connections to string theory. Furthermore, the τ function theory, which has been extensively utilized to explain the vibrational modes and interactions of strings in physics, provides an important link between integrable systems and string theory [34, 35].

Additionally, several aspects of the study of dispersionless integrable systems are drawing more attention including the long-time behaviour and the possible wave breaking properties of dispersionless equations. Recently, Manakov and Santini introduced a new IST (inverse scattering transform) method for solving the dispersionless integrable equations at the formal level, such as the dKP equation, the heavenly equation of Plebanski, the Pavlov equation and so on [36–41]. One key step in the Manakov-Santini method is to establish the relevant nonlinear Riemann-Hilbert problem with the dispersionless equation. Two important dispersionless integrable systems, the (3+1)-dimensional Dunajski hierarchy and the (2+1)-dimensional dDS system, are discussed by using the Manakov-Santini method [41, 42]. In [29], an alternate construction of the twistor structure, which provides general solutions for dispersionless systems, is provided by Takasaki and Takebe. This is a kind of Riemann-Hilbert construction. Originally, the Riemann-Hilbert problem was a series of equations pertaining to the properties of analytic functions. With the extensive study of integrable dynamical systems, it gradually associates with the integrable systems. Actually, in the study of classical integrable systems, we usually focus on the linearised Riemann-Hilbert problem, but in the exploration of dispersionless integrable systems, the relevant nonlinear Riemann-Hilbert problem draws more attention.

The paper is organized as follows. In Sec. 2, we construct the related hierarchy, which contains an infinite number of compatible partial differential equations. In Sec. 3, we show the existence of the τ function. In Sec. 4, we discuss the twistor structure of the integrable hierarchy. In Sec. 5, we consider the relevant nonlinear Riemann-Hilbert problem.

2. LAX FORMALISM AND HIERARCHY

In this Section, we construct a hierarchy associated with the dispersionless system by taking a truncation of the eigenfunction from the Lax pair of equation (1). This hierarchy contains an infinite number of integrable dispersionless equations, which are infinite symmetries of equation (1). In [7], the authors proposed several classes of integrable models, and the integrability of these dispersionless equations is reflected in the compatibility of the vector field Lax pair. In particular, the Hamiltonian vector field Lax pair of equation (1) reads as follows

$$\begin{aligned} L_1 &= \partial_t - \{H_1, \cdot\} = \partial_y - \lambda\partial_x + 2u_x\lambda\partial_\lambda, & (2)\\ L_2 &= \partial_y - \{H_2, \cdot\} = \partial_t + \left(\frac{1}{2}\lambda^2 + u\lambda\right)\partial_x - \left(u_x\lambda + u_y + 2uu_x\right)\lambda\partial_\lambda, & (3) \end{aligned}$$

in which Hamiltonians H_1 and H_2 are given by

$$\begin{aligned} H_1 &= \lambda + 2u, & (4)\\ H_2 &= -\left(\frac{1}{4}\lambda^2 + u\lambda + u^2 + \partial_x^{-1}u_y\right). & (5) \end{aligned}$$

Here and hereafter, in this paper, the symbol $\{,\}$ stands for the Poisson bracket in $2D$ phase space (λ, x) as follows

$$\{A,B\} = \lambda\frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial x} - \lambda\frac{\partial A}{\partial x}\frac{\partial B}{\partial \lambda}.$$

As we know, the dispersionless system (1) arises from the commutation condition

$$[L_1, L_2] = 0. \tag{6}$$

In fact, the commutation condition (6) is equivalent to the following Zakharov-Shabat equation

$$\frac{H_1}{\partial y} - \frac{H_2}{\partial t} + \{H_1, H_2\} = 0. \tag{7}$$

In order to constructe infinite symmetries for the dispersionless system (1), we first consider an eigenfunction of the vector field. With a given simple closed curve around the original point in the complex λ -plane, the eigenfunction Φ of L_1 can be described by the following formal Laurent expansion

$$\Phi = \lambda + \sum_{k\le 0} a_k \lambda^k. \tag{8}$$

In fact, its coefficients a_k can be calculated directly from $L_1\Phi = 0$, in which all coefficients rely on the function u and their derivatives or integrals with respect to the independent variables under x, y. For example,

$$\begin{aligned} a_0 &= 2u, \\ a_{-1} &= 2\partial_x^{-1} u_y, \\ a_{-2} &= 2\partial_x^{-2} u_{yy} + 4\partial_x^{-1}\left(u_x \partial_x^{-1} u_y\right). \end{aligned}$$

Then we can recursively work out the arbitrary coefficients of the eigenfunction Φ.

In constructing the hierarchy of the dispersionless system (1), we need to consider m powers of the eigenfunction Φ. Based on the coefficients a_k of the eigenfunction Φ obtained above, we denote Φ^m as follows

$$\Phi^m = \lambda^m + \sum_{k\le m-1} q_k^{(m)} \lambda^k, \quad m = 1, 2, \cdots,$$

in which $q_k^{(m)} = q_k^{(m)}(a_0, a_{-1}, \ldots, a_{k-1})$.

Meanwhile, according to the expression (8), conversely, the parameter λ can be formally represented by Φ as the following Laurent expansion

$$\lambda = \Phi + \sum_{k\le 0} p_k \Phi^k, \tag{9}$$

where $p_k = p_k(a_0, a_{-1}, \ldots, a_k)$.

Next, we start to construct infinite symmetries of the dispersionless system (1), which form the integrable hierarchy, and the Lax formalisms are vector fields, which all have Hamiltonian structures. More specifically these Hamiltonians are truncations of higher powers of the eigenfunction Φ. Furthermore, we prove that all vector fields share the same eigenfunctions.

Definition 2.1 *For any positive integer m, we define a series of the Hamiltonian vector fields as follows*

$$L_m = \partial_{t_m} - \{H_m, \cdot\}, \tag{10}$$

where

$$H_m = (-1)^{m-1}\left(\frac{1}{m^2}\Phi^m\right)_{\geq 0}, \tag{11}$$

and $()_{\geq 0}$ denotes the projection of the polynomial in λ with a positive power. In particular, we take $t_1 = y$, $t_2 = t$.

We now address the compatibility of the operator L_1 with this series of operators, which leads to the following conclusion.

Proposition 2.1 *The Zakharov-Shabat equations*

$$\frac{\partial H_1}{\partial t_m} - \frac{\partial H_m}{\partial y} + \{H_1, H_m\} = 0, \tag{12}$$

are equivalent to the following flow equations

$$2u_{t_m} = (-1)^{m-1} q_{0,y}^{(m)}. \tag{13}$$

Proof. In fact, the Zakharov-Shabat equations (12) represent

$$\begin{aligned}\frac{\partial H_1}{\partial t_m} &= L_1(H_m)\\ &= L_1\left[(-1)^{m-1}\left(\frac{1}{m^2}\Phi^m\right)_{\geq 0}\right]\\ &= \lambda + (-1)^{m-1} 2u_{t_m}.\end{aligned} \tag{14}$$

Since the eigenfunction space is a ring, then $(-1)^{m-1}\frac{1}{m^2}\Phi^m$ are also eigenfunctions and satisfy the equation $L_1((-1)^{m-1}\frac{1}{m^2}\Phi^m) = 0$. The following recursive relations are obtained as

$$\begin{aligned}&q_{l,y}^{(m)} - q_{l-1,x}^{(m)} + 2u_x l q_l^{(m)} = 0,\\ &q_{m-1}^{(m)} = \frac{2u}{m}, \quad q_l^{(m)} = 0, \ (l > m-1).\end{aligned} \tag{15}$$

Bringing the above recursive relations into the display formula (14), then gives the flow equations (13).

Obviously, the Zakharov-Shabat equations (12) are equivalent to the following commutation condition

$$[L_1, L_m] = 0. \tag{16}$$

As we know, the space of vector field eigenfunctions is a ring whose base consists of two independent eigenfunctions. Combining the above commutation (16), for any positive integers n, the vector fields L_m, L_n share the eigenfunction space with L_1. This implies that L_n satisfy the commutation condition with L_m. Thus we can define the hierarchy of the dispersionless system (1).

Theorem 2.1 *The Zakharov-Shabat equations*

$$\frac{\partial H_m}{\partial t_n} - \frac{\partial H_n}{\partial t_m} + \{H_m, H_n\} = 0, \quad m,n = 1,2,\cdots, \tag{17}$$

are equivalent to a hierarchy of compatible systems.

Proof. In fact, the Zakharov-Shabat equations (17) are equivalent to the commutation condition as follows

$$[L_m, L_n] = 0. \tag{18}$$

The proof is given below. For Zakharov-Shabat equations

$$\begin{aligned}
&\frac{\partial H_m}{\partial t_n} - \frac{\partial H_n}{\partial t_m} + \{H_m, H_n\} \\
&= (-1)^{m-1}\left(\frac{1}{m}\Phi^{m-1}\Phi_{t_n}\right)_{\geq 0} + (-1)^n\left(\frac{1}{n}\Phi^{n-1}\Phi_{t_m}\right)_{\geq 0} + \{H_m, H_n\} \\
&= \left[(-1)^{m-1}\frac{1}{m}\Phi^{m-1}\{H_n,\Phi\} + (-1)^n\frac{1}{n}\Phi^{n-1}\{H_m,\Phi\} + \{H_m,H_n\}\right]_{\geq 0} \\
&= \left[\left\{H_n, (-1)^{m-1}\frac{1}{m^2}\Phi^m\right\} - \left\{H_m, (-1)^{n-1}\frac{1}{n^2}\Phi^n\right\} + \{H_m,H_n\}\right]_{\geq 0} \\
&= \left[\left\{H_n + (-1)^n\frac{1}{n^2}\Phi^n, (-1)^{m-1}\frac{1}{m^2}\Phi^m\right\} - \left\{H_m, H_n + (-1)^n\frac{1}{n^2}\Phi^n\right\}\right]_{\geq 0} \\
&= \left[\left\{(H_n)_{<0}, (-1)^{m-1}\frac{1}{m^2}\Phi^m\right\} + \left\{H_m, (H_n)_{<0}\right\}\right]_{\geq 0} \\
&= \left[\left\{(H_m)_{<0}, (H_n)_{<0}\right\}\right]_{\geq 0},
\end{aligned}$$

whose value equals to 0.

On the other hand,

$$
\begin{aligned}
[L_m, L_n] &= L_m L_n - L_n L_m \\
&= (\partial_{t_m} - \{H_m, \cdot\})(\partial_{t_n} - \{H_n, \cdot\}) \\
&\quad - (\partial_{t_n} - \{H_n, \cdot\})(\partial_{t_m} - \{H_m, \cdot\}) \\
&= (\partial_{t_m} - \lambda H_{m,\lambda}\partial_x + \lambda H_{m,x}\partial_\lambda)(\partial_{t_n} - \lambda H_{n,\lambda}\partial_x + \lambda H_{n,x}\partial_\lambda) \\
&\quad - (\partial_{t_n} - \lambda H_{n,\lambda}\partial_x + \lambda H_{n,x}\partial_\lambda)(\partial_{t_m} - \lambda H_{m,\lambda}\partial_x + \lambda H_{m,x}\partial_\lambda) \\
&= \lambda \left(\frac{\partial H_m}{\partial t_n} - \frac{\partial H_n}{\partial t_m} + \{H_m, H_n\} \right)_\lambda \\
&\quad + \lambda \left(\frac{\partial H_n}{\partial t_m} - \frac{\partial H_m}{\partial t_n} + \{H_n, H_m\} \right)_x .
\end{aligned}
$$

Likewise, according to the Zakharov-Shabat equations (17), this value also equals to 0.

Example 2.1 *For the flow equations* (13)*, when* $m = 1, 2, 3,$ *we bring them in the recursive relations* (15)*,*

$$m = 1, \quad u_{t_1} = u_y, \tag{19}$$

$$m = 2, \quad 2u_{xt_2} + u_{yy} + (u^2)_{xy} = 0, \tag{20}$$

$$m = 3, \quad 3u_{xt_3} - 2(u^2)_{yy} - 2(u_x\partial_x^{-1}u_y)_y - 4(u^2u_x)_y - \partial_x^{-1}u_{yyy} = 0. \tag{21}$$

Equation (20) *is the commutation condition* $[L_1, L_2] = 0$ *arising from the Hamiltonian vector field Lax pair. Additionally, equation* (21) *is equivalent to* $[L_1, L_3] = 0$*.*

To give another example, by taking $m = 3$*, then the Hamiltonian* (11) *can be written as*

$$H_3 = \frac{1}{9}\lambda^3 + \frac{1}{3}a_0\lambda^2 + \frac{1}{3}a_0^2\lambda + \frac{1}{3}a_1\lambda + \frac{2}{3}a_0a_1 + \frac{1}{3}a_2 + \frac{1}{9}a_0^3.$$

The following nonlinear system

$$
\begin{aligned}
&3u_{t_3} + 8uu_{t_2} + 4u^2u_y + 2u\partial_x^{-1}u_{yy} + 2\partial_x^{-1}u_{yt_2} - 2u_y\partial_x^{-1}u_y = 0, \\
&u_{yt_3} + \left(u^2\right)_{xt_3} + 2\left(u^2\right)_{yt_2} + 4\left(u^2u_x\right)_{t_2} + \partial_x^{-1}u_{yyt_2} + 2\left(u_x\partial_x^{-1}u_y\right)_{t_2} = 0,
\end{aligned}
$$

derives from the Zakharov-Shabat equations (17)*, which equals to the commutation condition* $[L_2, L_3] = 0$*.*

3. THE TAU FUNCTION OF THE HIERARCHY

In this Section, we present two exterior differential 2-forms that are equivalent to the hierarchy. In addition to the eigenfunction Φ, we will construct another independent eigenfunction Ψ. Based on this, we obtain the S function and the existence of the significant τ function.

Definition 3.1 *We introduce an exterior differential 2-form*

$$\omega = \frac{d\lambda}{\lambda} \wedge dx + \sum_{n=1}^{\infty} dH_n \wedge dt_n, \tag{22}$$

where "d" denotes the full differentiation.

In fact, this exterior differential 2-form ω is obviously a closed form, hence,

$$d\omega = 0.$$

By the Zakharov-Shabat equations (17), ω satisfies the following relation

$$\omega \wedge \omega = 0.$$

The above two relations indicate the existence of two functions P and Q that give a pair of Darboux coordinates as

$$\omega = \frac{dP}{P} \wedge dQ.$$

Proposition 3.1 *Taking $P = \Phi$, there exists another eigenfunction Ψ satisfying*

$$\omega = \frac{d\Phi}{\Phi} \wedge d\Psi, \tag{23}$$

in which

$$\Phi = \lambda + \sum_{k \leq 0} a_k \lambda^k, \tag{24}$$

$$\Psi = x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t_n}{n} \Phi^n + \sum_{i=1}^{\infty} v_i \Phi^{-i}. \tag{25}$$

In particular,

$$\{\Phi, \Psi\} = \Phi,$$

and they are the eigenfunctions of the vector field L_m, satisfying $L_m(K) = 0$, in which $K = \Phi, \Psi$. Then the Lax expressions read as

$$\frac{\partial \Phi}{\partial t_n} = \{H_n, \Phi\}, \quad \frac{\partial \Psi}{\partial t_n} = \{H_n, \Psi\}, \tag{26}$$

and are equivalent to the hierarchy of the dispersionless system (1).

Next, we prove the equivalence of two forms of 2-form ω. Firstly, we show that the known Lax hierarchy derives the exterior differential equations. Looking back at the equation (22) and equation (23), they can both be written as a linear combination of $d\lambda \wedge dx$, $d\lambda \wedge dt_n$, $dx \wedge dt_n$, and $dt_m \wedge dt_n$. Obviously, there are coefficients of $d\lambda \wedge dx$ for

$$\frac{1}{\Phi} \begin{vmatrix} \Phi_\lambda & \Phi_x \\ \Psi_\lambda & \Psi_x \end{vmatrix} = \frac{1}{\lambda}.$$

Similarly, we have the following coefficients of $d\lambda \wedge dt_n$, $dx \wedge dt_n$, and $dt_m \wedge dt_n$ as

$$
\begin{aligned}
\frac{1}{\Phi}\begin{vmatrix}\Phi_\lambda & \Phi_{t_n}\\ \Psi_\lambda & \Psi_{t_n}\end{vmatrix} &= \frac{1}{\Phi}\left(\Phi_\lambda\left\{H_n,\Psi\right\} - \left\{H_n,\Phi\right\}\Psi_\lambda\right)\\
&= \frac{1}{\Phi}\left(H_{n,\lambda}\left\{\Phi,\Psi\right\}\right) = \frac{\partial H_n}{\partial \lambda},\\
\frac{1}{\Phi}\begin{vmatrix}\Phi_x & \Phi_{t_n}\\ \Psi_x & \Psi_{t_n}\end{vmatrix} &= \frac{1}{\Phi}\left(\Phi_x\left\{H_n,\Psi\right\} - \left\{H_n,\Phi\right\}\Psi_x\right)\\
&= \frac{1}{\Phi}\left(H_{n,x}\left\{\Phi,\Psi\right\}\right) = \frac{\partial H_n}{\partial x},\\
\frac{1}{\Phi}\begin{vmatrix}\Phi_{t_m} & \Phi_{t_n}\\ \Psi_{t_m} & \Psi_{t_n}\end{vmatrix} &= \frac{1}{\Phi}\left(\left\{H_m,\Phi\right\}\left\{H_n,\Psi\right\} - \left\{H_n,\Phi\right\}\left\{H_m,\Psi\right\}\right)\\
&= \frac{1}{\Phi}\left(\left\{\Phi,\Psi\right\}\left\{H_m,H_n\right\}\right) = \left\{H_m,H_n\right\}.
\end{aligned}
$$

Secondly, the Lax hierarchy is deduced in turn from the exterior differential equations. The coefficients of $d\lambda \wedge dx$ hold as

$$\Phi = \lambda\left(\Phi_\lambda \Psi_x - \Psi_\lambda \Phi_x\right) = \left\{\Phi,\Psi\right\}.$$

Comparing the coefficients on both sides of $d\lambda \wedge dt_n$, $dx \wedge dt_n$, there are

$$
\begin{aligned}
\frac{1}{\Phi}\left(\Phi_\lambda \Psi_{t_n} - \Psi_\lambda \Phi_{t_n}\right) = H_{n,\lambda},\\
\frac{1}{\Phi}\left(\Phi_{t_n} \Psi_x - \Psi_{t_n} \Phi_x\right) = H_{n,x}.
\end{aligned}
$$

For the above equations, we can accurately figure out Φ_{t_n} and Ψ_{t_n}. They are given by

$$
\begin{aligned}
\Phi_{t_n} = \lambda\left(H_{n,\lambda}\Phi_x - H_{n,x}\Phi_\lambda\right) = \left\{H_n,\Phi\right\},\\
\Psi_{t_n} = \lambda\left(H_{n,\lambda}\Psi_x - H_{n,x}\Psi_\lambda\right) = \left\{H_n,\Psi\right\}.
\end{aligned}
$$

The process of proving the equivalence of exterior differential equations is complete.

Combining the two equivalent definitions of the 2-form ω, the following equation holds

$$d\Psi \wedge \frac{d\Phi}{\Phi} + \frac{d\lambda}{\lambda} \wedge dx + \sum_{n=1}^{\infty} dH_n \wedge dt_n = 0.$$

This equation can also be written as

$$d\left(\Psi d\log\Phi + \log\lambda dx + \sum_{n=1}^{\infty} H_n dt_n\right) = 0.$$

This implies the existence of a function S such that

$$dS = \Psi d\log\Phi + \log\lambda dx + \sum_{n=1}^{\infty} H_n dt_n.$$

Then there are the following equations

$$\frac{\partial S}{\partial x} = \log\lambda, \quad \frac{\partial S}{\partial t_n} = H_n, \quad \frac{\partial S}{\partial \log\Phi} = \Psi. \tag{27}$$

In fact, even though S is not a true potential function, it has some profound implications in the hierarchy. Then S can be expressed in the form of the Laurent series about Φ, Ψ.

Proposition 3.2 *S is given by*

$$S = x\log\Phi + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t_n}{n^2}\Phi^n - \sum_{i=1}^{\infty} S_i\Phi^{-i}, \quad S_i = \frac{v_i}{i}. \tag{28}$$

Through the equations (27) *and* (28) *we show that H_n can be written in the form of a Laurent series related to S_i as follows*

$$H_n = \frac{\partial S}{\partial t_n} = (-1)^{n-1}\frac{1}{n^2}\Phi^n - \sum_{i=1}^{\infty}\frac{\partial S_i}{\partial t_n}\Phi^{-i}. \tag{29}$$

In particular, for the above expression of H_n, when $n = 1$, we have

$$H_1 = \Phi - \sum_{i=1}^{\infty}\frac{\partial S_i}{\partial y}\Phi^{-i},$$

which is also equivalent to equation (9).

Next, we introduce the 1-form residue operator

$$res\sum a_n\lambda^n = a_{-1}.$$

It has the following properties.

Lemma 3.1 *For any Laurent series L and M of λ,*

$$\begin{aligned} &res d_\lambda L = 0, \\ &res L d_\lambda M = -res M d_\lambda L, \\ &res L d_\lambda M = res\left(L_{\geq 0}\right) d_\lambda\left(M_{\leq -1}\right) + res\left(L_{\leq -1}\right) d_\lambda\left(M_{\geq 0}\right). \end{aligned}$$

Lemma 3.2 *For any positive integer n,*

$$res\Phi^{n-1}d_\lambda\Phi = \delta_{n,-1}.$$

From the above properties of the residue operator, we can obtain the following Lemma and prove it.

Lemma 3.3 *For the Laurent series Ψ, the partial derivatives of the coefficients v_i ,*

$$\frac{\partial v_i}{\partial t_n} = res\Phi^i d_\lambda H_n.$$

Proof. For the Laurent series Ψ, from the chain rule,

$$\frac{\partial \Psi}{\partial t_n} = (-1)^{n-1}\frac{1}{n}\Phi^n + \frac{\partial \Psi}{\partial \Phi}\frac{\partial \Phi}{\partial t_n} + \sum_{i=1}^{\infty}\frac{\partial v_i}{\partial t_n}\Phi^{-i},$$

in which

$$\frac{\partial \Psi}{\partial \Phi} = \sum_{n=1}^{\infty}(-1)^{n-1}t_n\Phi^{n-1} - \sum_{i=1}^{\infty} iv_i\Phi^{-i-1}.$$

Then

$$\begin{aligned}
\frac{\partial v_i}{\partial t_n} &= res\Phi^{i-1}\left(\frac{\partial \Psi}{\partial t_n} - \frac{\partial \Psi}{\partial \Phi}\frac{\partial \Phi}{\partial t_n}\right)d_\lambda\Phi \\
&= res\Phi^{i-1}\left(\{H_n,\Psi\} - \{H_n,\Phi\}\frac{\partial \Psi}{\partial \Phi}\right)d_\lambda\Phi \\
&= res\Phi^{i-1}\left[(\lambda H_{n,\lambda}\Psi_x - \lambda H_{n,x}\Psi_\lambda) - (\lambda H_{n,\lambda}\Phi_x - \lambda H_{n,x}\Phi_\lambda)\frac{\partial \Psi}{\partial \Phi}\right]d_\lambda\Phi \\
&= res\Phi^{i-1}\left[H_{n,\lambda}(\lambda\Psi_x\Phi_\lambda - \lambda\Phi_x\Psi_\lambda) - H_{n,x}(\lambda\Psi_\lambda\Phi_\lambda - \lambda\Phi_\lambda\Psi_\lambda)\right]d\lambda \\
&= res\Phi^{i-1}\left(\frac{\partial H_n}{\partial \lambda}\Phi\right)d\lambda \\
&= res\Phi^i d_\lambda H_n.
\end{aligned}$$

Based on the above preparation, we will give an existence Theorem for the τ function.

Theorem 3.1 *For the hierarchy of the dispersionless equation* (1)*, there exists the τ function satisfying*

$$d\log\tau = v_n dt_n,$$

in which "d" represents the differentiation of t_n.

Proof. In fact, to prove the existence of τ function, we just need to show that the right-hand side of the equation is in closed form. It turn out to be

$$\frac{\partial v_n}{\partial t_m} = \frac{\partial v_m}{\partial t_n}.$$

Then, according to the previous Lemma, the following results are available,

$$\begin{aligned}
\frac{\partial v_n}{\partial t_m}-\frac{\partial v_m}{\partial t_n} &= res\Phi^n d_\lambda H_m - res\Phi^m d_\lambda H_n \\
&= res\left((\Phi^n)_{\geq 0}+(\Phi^n)_{<0}\right) d_\lambda (\Phi^m)_{\geq 0} \\
&\quad -res\left((\Phi^m)_{\geq 0}+(\Phi^m)_{<0}\right) d_\lambda (\Phi^n)_{\geq 0} \\
&= res\,(\Phi^n)_{<0}\, d_\lambda (\Phi^m)_{\geq 0} - res\,(\Phi^m)_{<0}\, d_\lambda (\Phi^n)_{\geq 0} \\
&= res\,(\Phi^n)_{<0}\, d_\lambda (\Phi^m)_{\geq 0} + res\,(\Phi^n)_{\geq 0}\, d_\lambda (\Phi^m)_{<0} \\
&= res\Phi^n d_\lambda \Phi^m \\
&= resm\Phi^{m+n-1} d_\lambda \Phi \\
&= m\delta_{m+n,-1}.
\end{aligned}$$

Since m,n are positive integers, the above equation is vanished. Eventually, the existence of the τ function is proved.

In fact, the τ function occupies a very important place in dispersionless integrable systems, and it contains all the information about the structure of the dispersionless hirearchy.

4. THE TWISTOR STRUCTURE OF THE HIERARCHY

In this Section, we modify the eigenfunctions Φ and Ψ using the dressing function. Then we further construct the twistor structure of the hierarchy.

Proposition 4.1 *Let Φ and Ψ be solutions of the hierarchy, and there exists a dressing function φ satisfying the following equations*

$$\Phi = e^{ad\,\varphi}(\lambda), \quad \Psi = e^{ad\,\varphi}\left(x+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t_n}{n}\lambda^n\right), \tag{30}$$

$$\nabla_{t_n,\varphi}\varphi = -(-1)^{n-1}\left(\frac{1}{n^2}e^{ad\,\varphi}(\lambda^n)\right)_{<0}, \quad n=1,2,\cdots, \tag{31}$$

in which

$$\varphi(t)=\sum_{n=1}^{\infty}\varphi_n(t)\lambda^{-n}.$$

Indeed, observing the above results, we derive an alternative expression for the eigenfunctions, and one can apply such expression in the following proof of the twistor structure of the hierarchy.

Proposition 4.2 *If $\varphi(t)=\sum_{n=1}^{\infty}\varphi_n(t)\lambda^{-n}$ satisfies the equations of* (31)*, then Φ and Ψ defined by* (30) *are solutions of the hierarchy.*

Proof.

$$\begin{aligned}
\partial_{t_n}\Phi=\partial_{t_n}\left(e^{ad\,\varphi}(\lambda)\right) &= \left\{\nabla_{t_n,\varphi}\varphi, e^{ad\,\varphi}(\lambda)\right\}\\
&= e^{e^{ad\,\varphi}}\left\{e^{-ad\,\varphi}\,\nabla_{t_n,\varphi}\varphi,\lambda\right\}\\
&= e^{ad\,\varphi}\left\{e^{-ad\,\varphi}\left((-1)^n\frac{1}{n^2}e^{ad\,\varphi}(\lambda^n)\right)_{<0},\lambda\right\}\\
&= e^{ad\,\varphi}\left\{e^{-ad\,\varphi}H_n,\lambda\right\}\\
&= \{H_n,\Phi\}.
\end{aligned}$$

Another Lax equation reads as

$$\begin{aligned}
\partial_{t_n}(\Psi) &= \partial_{t_n}\left(e^{ad\,\varphi}\left(x+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t_n}{n}\lambda^n\right)\right)\\
&= \{\nabla_{t_n,\varphi}\varphi,\Psi\}+e^{ad\,\varphi}\left((-1)^{n-1}\frac{1}{n}\lambda^n\right)\\
&= \left\{H_n+(-1)^n\frac{1}{n^2}e^{ad\,\varphi}(\lambda^n),\Psi\right\}+e^{ad\,\varphi}\left((-1)^{n-1}\frac{1}{n}\lambda^n\right)\\
&= e^{ad\,\varphi}\left(\left\{e^{-ad\,\varphi}H_n+(-1)^n\frac{1}{n^2}\lambda^n,x+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t_n}{n}\lambda^n\right\}\right)\\
&\quad +e^{ad\,\varphi}\left((-1)^{n-1}\frac{1}{n}\lambda^n\right)\\
&= e^{ad\,\varphi}\left(\left\{e^{-ad\,\varphi}H_n,x+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t_n}{n}\lambda^n\right\}\right)\\
&= \{H_n,\Psi\}.
\end{aligned}$$

Definition 4.1 *Let a pair of two functions $(f(\lambda,x),g(\lambda,x))$ and a pair of Laurent series (Φ,Ψ), where Φ and Ψ are forms of equations* (24) *and* (25) *respectively. Suppose that they satisfy the canonical Poisson relation $\{f(\lambda,x),g(\lambda,x)\}=f(\lambda,x)$; $F=f(\Phi,\Psi)$ and $G=g(\Phi,\Psi)$ are both Taylor in λ as*

$$(f(\Phi,\Psi))_{\leq -1}=0,\quad (g(\Phi,\Psi))_{\leq -1}=0.$$

Then the pair (Φ,Ψ) is a solution of the hierarchy and for this reason, (f,g) is the twistor structure of this solution.

Theorem 4.1 *The hierarchy with twistor structure in their solutions, i.e., if (Φ,Ψ) is a solution to the dispersionless equation* (1)*, then there exists the twistor structure (f,g) of this hierarchy.*

Proof. First, we show that $\{f,g\} = f$. Let $e^{ad\,\varphi(\vec{t})}$ be the dressing operator corresponding to (Φ,Ψ). When $\vec{t} = \vec{0}$, $\vec{t} = (t_1, t_2, \cdots)$. We set

$$f(\lambda,x) = e^{-ad\,\varphi}|_{\vec{t}=\vec{0}}\,\lambda,\quad g(\lambda,x) = e^{-ad\,\varphi}|_{\vec{t}=\vec{0}}\,x. \tag{32}$$

In fact,

$$\begin{aligned}\{f(\lambda,x), g(\lambda,x)\} &= e^{-ad\,\varphi}|_{\vec{t}=\vec{0}}\{\lambda,x\}\\ &= e^{-ad\,\varphi}|_{\vec{t}=\vec{0}}\lambda\\ &= f(\lambda,x).\end{aligned}$$

For equation (30),

$$\Phi|_{\vec{t}=\vec{0}} = e^{ad\,\varphi}|_{\vec{t}=\vec{0}}\,\lambda,\quad \Psi|_{\vec{t}=\vec{0}} = e^{ad\,\varphi}|_{\vec{t}=\vec{0}}\,x.$$

Then, the following relations are obtained from equation (32),

$$\begin{aligned} f\left(\Phi|_{\vec{t}=\vec{0}}, \Psi|_{\vec{t}=\vec{0}}\right) &= e^{ad\,\varphi}|_{\vec{t}=\vec{0}} f(\lambda,x) = \lambda,\\ g\left(\Phi|_{\vec{t}=\vec{0}}, \Psi|_{\vec{t}=\vec{0}}\right) &= e^{ad\,\varphi}|_{\vec{t}=\vec{0}} g(\lambda,x) = x.\end{aligned} \tag{33}$$

As a result,

$$\begin{aligned}\{f(\Phi,\Psi), g(\Phi,\Psi)\} &= e^{ad\,\varphi}|_{\vec{t}=\vec{0}}\{f(\lambda,x), g(\lambda,x)\}\\ &= e^{ad\,\varphi}|_{\vec{t}=\vec{0}} f(\lambda,x)\\ &= f(\Phi,\Psi).\end{aligned}$$

Secondly, we prove that $(f(\Phi,\Psi))_{\leq -1} = 0$. Since Φ and Ψ satisfy the Lax equations (26), then

$$\begin{aligned}\frac{\partial f(\Phi,\Psi)}{\partial t_n} &= f_\Phi(\Phi,\Psi)\cdot\Phi_{t_n} + f_\Psi(\Phi,\Psi)\cdot\Psi_{t_n}\\ &= f_\Phi(\Phi,\Psi)\{H_n,\Phi\} + f_\Psi(\Phi,\Psi)\{H_n,\Psi\}\\ &= \{H_n, f(\Phi,\Psi)\}.\end{aligned} \tag{34}$$

When $\vec{t} = \vec{0}$, based on the equations (33) and (34), the result is as follows

$$\begin{aligned}\frac{\partial f(\Phi,\Psi)}{\partial t_n}|_{\vec{t}=\vec{0}} &= \{H_n, f(\Phi=0,\Psi=0)\}\\ &= \{H_n,\lambda\}\\ &= -\lambda H_{n,x}.\end{aligned}$$

Obviously, the above equation does not contain the negative part of λ. Beyond that, for equation $(\partial/\partial_t)^\alpha f(\Phi,\Psi)|_{(\vec{t}=\vec{0})}$, regardless of the value of α, the Taylor expansion coefficients of this equation at $\vec{t} = \vec{0}$ do not contain negative terms of λ. Ultimately we can prove that $(f(\Phi,\Psi))_{\leq -1} = 0$. The same reasoning leads to $(g(\Phi,\Psi))_{\leq -1} = 0$.

5. THE RELEVANT NONLINEAR RIEMANN-HILBERT PROBLEM FOR CONSTRUCTING SOLUTIONS

The longtime behaviour and the possible wave breaking properties are important aspects in the study of the dispersionless equations. First and the key step is to relate the dispersionless equations to the nonlinear Riemann-Hilbert problem using the Manakov-Santini method. In fact, solving equation (1) can be transformed into studying the nonlinear Riemann-Hilbert problem. The result is shown in the following Theorem.

Theorem 5.1 *Consider the vector nonlinear Riemann-Hilbert problem on the real line as*

$$\vec{\pi}^{+}(\lambda) = \vec{R}(\vec{\pi}^{-}(\lambda)), \quad \lambda \in \mathbb{R}, \tag{35}$$

in which $\vec{\pi}^{+}(\lambda)$, $\vec{\pi}^{-}(\lambda) \in \mathbb{C}^2$ *are two-dimensional vector functions resolved in the upper and lower halves of the complex* λ *plane, respectively, and normalized them into the following form as*

$$\vec{\pi}^{\pm}(\lambda) = \begin{pmatrix} \lambda + 2u \\ -\frac{t}{2}\lambda^2 - 2ut\lambda + y\lambda + x + 2uy - 2u^2 t - 2t\partial_x^{-1} u_y \end{pmatrix} + o(\lambda^{-1}), \tag{36}$$

in which $|\lambda| \gg 1$.

In fact,

$$\partial_x^{-1} u = \lim_{\lambda\to\infty} \frac{\lambda}{2}\left(\pi_2^{\pm}(\lambda) - y\pi_1^{\pm}(\lambda) + \frac{t}{2}\pi_1^{\pm^2}(\lambda) - x\right), \tag{37}$$

and the differentiable spectral datum $\vec{R}(\vec{\xi}) = (R_1(\xi_1,\xi_2), R_2(\xi_1,\xi_2))$*, the vector* $\vec{\xi} \in \mathbb{C}^2$, $\lambda \in \mathbb{R}$*, satisfies the following given realistic constraint as*

$$\vec{R}(\overline{\vec{R}(\bar{\vec{\xi}})}) = \vec{\xi}. \tag{38}$$

Applying the vector fields Lax pair L_1, L_2 *defined by equations* (2) *and* (3) *on the above nonlinear Riemann-Hilbert problem* (35)*, there are* $L_j\vec{\pi}^{+}(\lambda) = JL_j\vec{\pi}^{-}(\lambda)$, $j = 1,2$*, where* J *satisfies the Jacobian matrix* $J_{mn} = \partial R_m/\partial\xi_n$, $(m,n = 1,2)$*. Then, assuming that both forms of the solution are unique,* $\vec{\pi}^{\pm}(\lambda)$ *of the Riemann-Hilbert problem* (35) *are the common eigenfunctions of the vector fields, i.e.,* $L_j\vec{\pi}^{\pm}(\lambda) = 0$ *and* u *is the solution to the equation* (1)*. Combining* $\vec{R}(\vec{\xi})$ *with constraint* (38)*, then the solution* u *is real, i.e.,* $u \in \mathbb{R}$.

Proof. Let the vector fields Lax pair L_1 and L_2 act on $\vec{\pi}^{\pm}(\lambda)$ in equation (36). In fact, when $\lambda \to \infty$, as a result,

$$L_j\vec{\pi}^{\pm}(\lambda) \to 0, \quad j = 1,2.$$

We are based on the nonlinear Riemann-Hilbert problem linearized version

$$L_j\vec{\pi}^{+}(\lambda) = JL_j\vec{\pi}^{-}(\lambda), \quad j = 1,2,$$

where J is the Jacobi matrix satisfying

$$J = \begin{pmatrix} \dfrac{\partial R_1}{\partial \xi_1} & \dfrac{\partial R_2}{\partial \xi_1} \\ \dfrac{\partial R_1}{\partial \xi_2} & \dfrac{\partial R_2}{\partial \xi_2} \end{pmatrix}.$$

Obviously, combining equation (35), the vectors $L_j\vec{\pi}^{\pm}(\lambda)$ solve the linearised Riemann-Hilbert problem. By uniqueness, we infer that $\vec{\pi}^{\pm}(\lambda)$ are shared eigenfunctions of the Lax pair L_j, obtaining

$$L_j\vec{\pi}^{\pm}(\lambda) = 0, \quad j = 1,2.$$

This indicates that the $\vec{\pi}^{\pm}(\lambda)$ are the solutions of the nonlinear Riemann-Hilbert problem. From the equation (36), one has

$$\partial_x^{-1}u = \lim_{\lambda\to\infty}\frac{\lambda}{2}\left(\pi_2^{\pm}(\lambda) - y\pi_1^{\pm}(\lambda) + \frac{t}{2}\pi_1^{\pm^2}(\lambda) - x\right).$$

Actually, it is the solution to the dispersionless system (1).

Similarly, by uniqueness, combining the constraint (38) of $\vec{R}(\vec{\xi})$ and the nonlinear Riemann-Hilbert problem (35), we can get the following expression

$$\overline{\vec{\pi}^{+}}(\lambda) = \vec{\pi}^{-}(\lambda).$$

It follows that the solution u is real, *i.e.*, $u \in \mathbb{R}$.

Afterwards, on the basis of the Manakov-Santini method, we will continue to delve into the longtime behavior of the dispersionless system (1) in our next research phase.

Acknowledgements. This work is supported by the National Natural Science Foundation of China under Grant Nos. 12271136, 12171133, and 12171132.

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