FRACTIONAL CHEBYSHEV SPECTRAL SCHEME FOR TIME-FRACTIONAL SEVENTH-ORDER KDV EQUATIONS WITH NON-SMOOTH SOLUTIONS

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July 28, 2025

One of the key challenges in solving time-fractional evolution equations, such as the seventh-order KdV equations, is the non-smooth behavior of the solution in the time direction, which arises from the weakly singular kernel of the fractional derivative. This non-smoothness often limits the performance of standard polynomial-based methods in accurately capturing the solution near the initial time. To address this, we adopt a fully spectral collocation method in both space and time, which offers high-order accuracy. In particular, we employ fractional-order Chebyshev functions as temporal basis functions, which are well-suited for approximating the singular behavior of fractional-order solutions and provide improved resolution in the early stages of the solution profile. Shifted Chebyshev polynomials are used in the spatial direction to maintain spectral accuracy. Our method transforms the original problem into a system of algebraic equations using an operational matrix approach, and the resulting numerical scheme is validated through comparisons with other existing numerical methods, demonstrating the accuracy and the superiority of the proposed spectral framework.

Key words: Fractional-order Chebyshev function, Seventh-order KdV equation, Spectral collocation method, Operational matrix.

1. INTRODUCTION

The time-fractional seventh-order Korteweg-de Vries (S-KdV) equation represents an important advancement in the modeling of nonlinear wave phenomena by extending the classical KdV framework through the introduction of fractional time derivatives combined with seventh-order spatial derivatives. This model effectively captures complex physical behaviors such as wave propagation with fine-scale oscillations and steep solitary waves, which are not adequately described by lower-order models [1–3]. The fractional time derivative introduces a memory effect, reflecting the influence of past states on the present dynamics, which is critical in modeling anomalous diffusion and nonlocal interactions in various physical, biological, and engineering systems [4–11].

From this point, studying the properties of different kinds of time-fractional differential equations and searching for efficient analytical and numerical solutions have become the focus of many recent works [12, 13]. Ali *et al.* [14] considered the time-fractional S-KdV equation and employed the Elzaki transformation, Adomian decomposition method, and homotopy perturbation technique to obtain analytical solutions. Ahmad and Saifullah [15] introduced series-form solutions of the time-fractional S-KdV equation using the ZZ-transform combined with the homotopy perturbation method, also applying this technique to related fractional KdV-type equations such as the time-fractional Lax and Sawada-Kotera-Ito equations. Bhrawy *et al.* [16] developed a space-time spectral collocation method based on double Jacobi polynomials and operational matrices of fractional derivatives to convert the problem into a system of algebraic equations, successfully solving time-fractional generalized Hirota-Satsuma coupled KdV systems. Qayyum *et al.* [17] presented traveling wave solutions of time-fractional Sawada-Kotera-Ito, Kaup-Kuperschimdt, and Lax KdV equations using a hybrid approach based on the homotopy perturbation method and Laplace transform. Akinyemi *et al.* [18] investigated two numerical methods for time-fractional Lax and Sawada-Kotera-Ito equations, specifically employing the q-homotopy analysis transform and fractional reduced differential transform techniques. Recently, for seventh-order KdV equations, the authors in [19] introduced a numerical scheme using a finite difference method in time, approximating the solution spatially with quintic B-spline functions. Other studies have examined different cases of time-fractional seventh-order KdV equations, see [20–22].

Despite these advances, the numerical solution of time-fractional S-KdV equations poses several challenges. One major difficulty arises from the non-smooth nature of solutions in the time variable caused by the weakly singular kernel of the fractional derivative. This non-smoothness often manifests as reduced regularity near the initial time, hindering the convergence and accuracy of conventional polynomial-based numerical methods. Traditional spectral methods, while highly accurate for smooth problems, typically struggle to approximate the fractional time derivative accurately when the solution exhibits singular or weakly regular behavior. This results in a demand for specialized basis functions or numerical techniques that can inherently accommodate such singularities [23].

To overcome these challenges, the current work employs fractional-order Chebyshev functions as the basis for approximating time-fractional derivatives. These fractional Chebyshev functions are specifically used to reflect the fractional order of differentiation and to capture the solution's singular behavior near the initial time effectively. Furthermore, by adopting a fully spectral method in both space and time, the proposed approach harnesses the spectral convergence properties and provides a high-order accurate solution, which is especially advantageous compared to semi-discrete or finite difference methods.

The paper is organized as follows. Section 2 presents important preliminaries on Chebyshev polynomials and fractional-order Chebyshev functions. Section 3 introduces the proposed numerical scheme for solving the time-fractional S-KdV equation, based on a spectral collocation approach combined with the operational matrix technique. Section 4 demonstrates the effectiveness of the numerical method through a representative example, including comparisons with existing numerical methods from the literature. Finally, Sec. 5 provides concluding remarks and discusses possible directions for future research.

2. FRACTIONAL CHEBYSHEV FUNCTIONS

Let $\mathcal{T}_k^L(x)$, $k \geq 0$, $x \in \Lambda = [0,L]$, denote the shifted Chebyshev polynomials with explicit representation:

$$\mathcal{T}_k^L(x) = \sum_{m=0}^{k} c_{k,m} \left(\frac{x}{L}\right)^m, \quad c_{k,m} = (-1)^{k-m} \frac{k(k+m-1)!2^{2m}}{(k-m)!(2m)!}. \tag{1}$$

These polynomials satisfy the orthogonality relation:

$$\int_0^L \mathcal{T}_k^L(x)\mathcal{T}_p^L(x) w_L(x) dx = \delta_{k,p} h_k, \tag{2}$$

with weight function and norm:

$$w_L(x) = \frac{1}{\sqrt{Lx - x^2}}, \quad h_k = \begin{cases} \pi, & k = 0, \\ \pi/2, & k \geq 1. \end{cases} \tag{3}$$

Any function $f \in L^2_{w_L}(\Lambda)$ admits the expansion:

$$f(x) = \sum_{k=0}^{\infty} f_k \mathcal{T}_k^L(x), \quad f_k = \frac{1}{h_k} \int_0^L \frac{f(x)\mathcal{T}_k^L(x)}{\sqrt{Lx - x^2}} dx. \tag{4}$$

Define the polynomial space:

$$\mathcal{P}_N = \text{span}\{\mathcal{T}_k^L(x) : 0 \leq k \leq N\}.$$

The orthogonal projection operator $\Pi_N^L : L^2_{w_L}(\Lambda) \to \mathcal{P}_N$ is defined by:

$$\begin{cases} \Pi_N^L f = f_N(x) = \sum_{k=0}^{N} f_k \mathcal{T}_k^L(x) = \mathbf{f}^\top \mathbf{T}_N(x), \\ \mathbf{f} = [f_0, \ldots, f_N]^\top, \quad \mathbf{T}_N(x) = [\mathcal{T}_0^L(x), \ldots, \mathcal{T}_N^L(x)]^\top. \end{cases} \tag{5}$$

For temporal discretization, we define fractional Chebyshev functions $\mathcal{F}_n^\lambda(t)$, $n \geq 0$, $t \in [0,1]$, $\lambda \in (0,1]$:

$$\mathcal{F}_n^\lambda(t) = \mathcal{T}_n(2t^\lambda - 1) = \sum_{m=0}^{n} c_{n,m} t^{m\lambda}. \tag{6}$$

These functions satisfy the orthogonality:

$$\int_0^1 \mathcal{F}_n^\lambda(t)\mathcal{F}_m^\lambda(t)w^\lambda(t)dt = \delta_{n,m}h_n^\lambda, \tag{7}$$

with fractional weight and norm:

$$w^\lambda(t) = \frac{t^{\lambda/2-1}}{\sqrt{1-t^\lambda}}, \quad h_n^\lambda = \frac{h_n}{\lambda}. \tag{8}$$

For $g \in L^2_{w^\lambda}([0,1])$, we have the expansion:

$$g(t) = \sum_{n=0}^{\infty} g_n \mathcal{F}_n^\lambda(t), \quad g_n = \frac{1}{h_n^\lambda}\int_0^1 g(t)\mathcal{F}_n^\lambda(t)w^\lambda(t)dt. \tag{9}$$

Define the fractional function space:

$$\mathcal{Q}_M = \text{span}\{\mathcal{F}_n^\lambda(t) : 0 \le n \le M\}.$$

The temporal projection operator $\Pi_M^\lambda : L^2_{w^\lambda} \to \mathcal{Q}_M$ is:

$$\begin{cases} \Pi_M^\lambda g = g_M(t) = \sum_{n=0}^{M} g_n \mathcal{F}_n^\lambda(t) = \mathbf{g}^\top \mathbf{F}_M(t), \\ \mathbf{g} = [g_0, \ldots, g_M]^\top, \quad \mathbf{F}_M(t) = [\mathcal{F}_0^\lambda(t), \ldots, \mathcal{F}_M^\lambda(t)]^\top. \end{cases} \tag{10}$$

3. NUMERICAL SCHEME

This Section presents the numerical solution of the time-fractional KdV equation:

$${}_C D_{0,t}^\sigma u + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} + C\frac{\partial^5 u}{\partial x^5} + D\frac{\partial^7 u}{\partial x^7} = g(x,t), \quad x \in \Lambda,\ t \in \mathcal{I}, \tag{11}$$

subject to:

$$u(x,0) = a(x), \quad x \in \Lambda,$$

$$u(0,t) = b_0(t), \quad \frac{\partial u(0,t)}{\partial x} = b_1(t), \quad \frac{\partial^2 u(0,t)}{\partial x^2} = b_2(t), \quad t \in \mathcal{I},$$

$$u(L,t) = c_0(t), \quad \frac{\partial u(L,t)}{\partial x} = c_1(t), \quad \frac{\partial^2 u(L,t)}{\partial x^2} = c_2(t), \quad \frac{\partial^3 u(L,t)}{\partial x^3} = c_3(t), \quad t \in \mathcal{I},$$

where $0 < \sigma \le 1$, A, B, C, D are known real constants, and $g(x,t)$, $a(x)$, $b_0(t)$, $b_1(t)$, $b_2(t)$, $c_0(t)$, $c_1(t)$, $c_2(t)$, $c_3(t)$ are known real functions.

The spectral solution is to find $u_{N,M} \in \mathcal{P}_N \otimes \mathcal{Q}_M$, $N \ge 7$, and $M \ge 1$ such that:

$$\begin{aligned} {}_C D_{0,t}^\sigma u_{N,M} &+ A u_{N,M}\partial_x u_{N,M} + B\partial_x^3 u_{N,M} \\ &+ C\partial_x^5 u_{N,M} + D\partial_x^7 u_{N,M} = g_{N,M}(x,t), \end{aligned} \tag{12}$$

where $g_{N,M} = \Pi_M^\lambda \Pi_N^L g$ is the projection of the source term:

$$g_{N,M}(x,t) = \mathbf{F}_M^\top(t)\mathbf{G}\mathbf{T}_N(x), \tag{13}$$

with coefficient matrix $\mathbf{G} = [G_{nm}]$ given by:

$$G_{nm} = \frac{1}{h_n h_m^\lambda} \int_0^1 \int_0^L w^\lambda(t) w_L(x) g(x,t) \mathcal{T}_n^L(x) \mathcal{F}_m^\lambda(t) dx dt.$$

The approximate solution is represented as:

$$u_{N,M}(x,t) = \mathbf{F}_M^\top(t)\mathbf{U}\mathbf{T}_N(x), \tag{14}$$

where $\mathbf{U} = [U_{nm}]$ is the $(N+1)\times(M+1)$ coefficient matrix. Spatial and temporal derivatives are computed using operational matrices:

$$\begin{aligned} \partial_x^k u_{N,M} &= \mathbf{F}_M^\top(t)\mathbf{U}\mathbf{D}^{(k)}\mathbf{T}_N(x), \\ {}_C D_{0,t}^\sigma u_{N,M} &= \mathbf{F}_M^\top(t)\mathbf{\Xi}_{(\sigma)}^\top \mathbf{U}\mathbf{T}_N(x). \end{aligned} \tag{15}$$

Theorem 3.1 *[24] The k-th derivative matrix for shifted Chebyshev polynomials satisfies:*

$$\frac{d^k}{dx^k}\mathbf{T}_N(x) = \mathbf{D}^{(k)}\mathbf{T}_N(x). \tag{16}$$

where $\mathbf{D}^{(k)} = (\mathbf{D}^{(1)})^k$ with elements:

$$d_{ij}^{(1)} = \begin{cases} \frac{2i}{L}, & j = 0, \\ \frac{4i}{L}, & j \geq 1,\ i = j+\ell,\ \ell \ odd \leq N, \\ 0, & otherwise. \end{cases}$$

Theorem 3.2 *[25] The Caputo derivative matrix for fractional Chebyshev functions is:*

$${}_C D_{0,t}^\sigma \mathbf{F}_M(t) = \mathbf{\Xi}_{(\sigma)}\mathbf{F}_M(t), \tag{17}$$

with elements:

$$\xi_{ij}^\sigma = \sum_{p=0}^{i}\sum_{q=0}^{j} \frac{\sqrt{\pi} c_{i,p} c_{j,q} \Gamma(p\lambda+1)\Gamma(p+q-\sigma/\lambda+1/2)}{\lambda h_j^\lambda \Gamma(p\lambda-\sigma+1)\Gamma(p+q-\sigma/\lambda+1)}.$$

The residual $\mathcal{R}_{N,M}$ is obtained by substituting derivatives into (12):

$$\begin{aligned} \mathcal{R}_{N,M}(x,t) =& \mathbf{F}_M^\top(t)\mathbf{\Xi}_{(\sigma)}^\top \mathbf{U}\mathbf{T}_N(x) + A(\mathbf{F}_M^\top(t)\mathbf{U}\mathbf{T}_N(x))(\mathbf{F}_M^\top(t)\mathbf{U}\mathbf{D}^{(1)}\mathbf{T}_N(x)) \\ &+ B\mathbf{F}_M^\top(t)\mathbf{U}\mathbf{D}^{(3)}\mathbf{T}_N(x) + C\mathbf{F}_M^\top(t)\mathbf{U}\mathbf{D}^{(5)}\mathbf{T}_N(x) \\ &+ D\mathbf{F}_M^\top(t)\mathbf{U}\mathbf{D}^{(7)}\mathbf{T}_N(x) - \mathbf{F}_M^\top(t)\mathbf{G}\mathbf{T}_N(x). \end{aligned}$$

Let $\{x_k\}_{k=0}^N$ and $\{t_m\}_{m=0}^M$ be roots of $\mathcal{T}_{N+1}^L(x)$ and $\mathcal{F}_{M+1}^\lambda(t)$, respectively. The collocation system is:

$$\begin{aligned}
&\mathcal{R}_{N,M}(x_k,t_m)=0, && 3\le k\le N-4,\, 1\le m\le M,\\
&\mathbf{F}_M^{\top}(0)\mathbf{U}\mathbf{T}_N(x_k)=a(x_k), && 0\le k\le N,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{T}_N(0)=b_0(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{D}^{(1)}\mathbf{T}_N(0)=b_1(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{D}^{(2)}\mathbf{T}_N(0)=b_2(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{T}_N(L)=c_0(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{D}^{(1)}\mathbf{T}_N(L)=c_1(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{D}^{(2)}\mathbf{T}_N(L)=c_2(t_m), && 1\le m\le M,\\
&\mathbf{F}_M^{\top}(t_m)\mathbf{U}\mathbf{D}^{(3)}\mathbf{T}_N(L)=c_3(t_m), && 1\le m\le M.
\end{aligned}\tag{18}$$

4. NUMERICAL RESULTS

In this Section, we assess the accuracy and effectiveness of the proposed numerical scheme by comparing its performance with the Legendre wavelet-based fractional operational matrix method (LWFOM) presented in [26].

We consider the following time-fractional S-KdV problem as a benchmark test case:

$$\begin{aligned}
&{}_C D_{0,t}^{\sigma} u + u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}+\frac{\partial^7 u}{\partial x^7}=\frac{t^{\sigma}\cos(x)}{\Gamma(1+\sigma)}-\frac{t^{4\sigma}\sin(x)\cos(x)}{(\Gamma(1+2\sigma))^2}+\frac{3t^{2\sigma}\sin(x)}{\Gamma(1+2\sigma)},\\
&u(x,0)=0,\qquad 0\le x\le 1.
\end{aligned}\tag{19}$$

The exact (non-smooth) solution of this equation is given by:

$$u(x,t)=\frac{t^{2\sigma}\cos(x)}{\Gamma(1+2\sigma)}.$$

Ray and Gupta [26] tackled this problem using the LWFOM method, where two-dimensional Legendre wavelet basis functions were employed to construct operational matrices for fractional integration and differentiation, thereby reducing the original fractional PDE into a system of algebraic equations.

Here, we apply our numerical scheme with $N=8$ and $M=3$ to solve this problem and compare our results with those given using the LWFOM at different values of σ.

Table 1 compares L_∞-errors and L_2-errors of $u_{N,M}(x,t)$ with $\lambda=1$ against those given by LWFOM [26] with $\sigma=1$.

Table 1.

Comparing L_2-errors and L_∞-errors of $u_{N,M}(x,t)$ *versus* the LWFOM [26] at $\sigma = 1$

	LWFOM [26] ($M = 8,\ k = 2$)		LWFOM [26] ($M = 6,\ k = 2$)		Present Scheme ($N = 8, M = 3$)	
t	l_2	L_∞	l_2	L_∞	l_2	L_∞
0.05	9.1×10^{-6}	5.3×10^{-6}	4.1×10^{-5}	2.1×10^{-5}	1.5×10^{-11}	1.8×10^{-11}
0.10	1.0×10^{-4}	5.5×10^{-5}	3.6×10^{-4}	1.8×10^{-4}	6.0×10^{-11}	7.4×10^{-11}
0.15	4.6×10^{-4}	2.3×10^{-4}	1.3×10^{-3}	6.4×10^{-4}	1.3×10^{-10}	1.6×10^{-10}
0.20	1.4×10^{-3}	6.9×10^{-4}	3.3×10^{-3}	1.5×10^{-3}	2.4×10^{-10}	2.9×10^{-10}
0.25	3.5×10^{-3}	1.6×10^{-3}	7.0×10^{-3}	3.2×10^{-3}	3.7×10^{-10}	4.6×10^{-10}
0.30	7.3×10^{-3}	3.2×10^{-3}	1.2×10^{-2}	5.8×10^{-3}	5.4×10^{-10}	6.6×10^{-10}
0.35	1.3×10^{-2}	5.8×10^{-3}	2.1×10^{-2}	9.6×10^{-3}	7.4×10^{-10}	9.0×10^{-10}
0.40	2.4×10^{-2}	9.9×10^{-3}	3.4×10^{-2}	1.5×10^{-2}	9.7×10^{-10}	1.1×10^{-09}
0.45	4.1×10^{-2}	1.6×10^{-2}	5.3×10^{-2}	2.2×10^{-2}	1.2×10^{-09}	1.5×10^{-09}
0.50	4.8×10^{-2}	2.6×10^{-2}	8.0×10^{-2}	3.3×10^{-2}	1.5×10^{-09}	1.8×10^{-09}

Table 2.

Comparing absolute errors of $u_{N,M}(x,t)$ *versus* the LWFOM [26] at $\sigma = 0.75$

	LWFOM [26]		Present Scheme $\lambda = 1$		Present Scheme $\lambda = 0.75$	
x	$t = 0.1$	$t = 0.2$	$t = 0.1$	$t = 0.2$	$t = 0.1$	$t = 0.2$
0.1	1.0×10^{-3}	6.8×10^{-3}	4.2×10^{-3}	2.4×10^{-3}	4.4×10^{-12}	6.0×10^{-11}
0.2	8.0×10^{-4}	7.8×10^{-3}	4.1×10^{-3}	2.3×10^{-3}	2.3×10^{-11}	3.2×10^{-10}
0.3	5.7×10^{-4}	8.8×10^{-3}	4.0×10^{-3}	2.3×10^{-3}	5.0×10^{-11}	6.8×10^{-10}
0.4	3.4×10^{-4}	9.7×10^{-3}	3.9×10^{-3}	2.2×10^{-3}	7.0×10^{-11}	9.4×10^{-10}
0.5	5.3×10^{-3}	1.4×10^{-2}	3.7×10^{-3}	2.1×10^{-3}	7.2×10^{-11}	9.7×10^{-10}
0.6	5.9×10^{-3}	1.5×10^{-2}	3.5×10^{-3}	2.0×10^{-3}	5.6×10^{-11}	7.6×10^{-10}
0.7	6.4×10^{-3}	1.6×10^{-2}	3.2×10^{-3}	1.8×10^{-3}	3.1×10^{-11}	4.2×10^{-10}
0.8	6.9×10^{-3}	1.7×10^{-2}	2.9×10^{-3}	1.6×10^{-3}	1.0×10^{-11}	1.3×10^{-10}
0.9	7.3×10^{-3}	1.8×10^{-2}	2.6×10^{-3}	1.5×10^{-3}	1.0×10^{-12}	1.3×10^{-11}
1.0	7.6×10^{-3}	1.8×10^{-2}	2.3×10^{-3}	1.3×10^{-3}	1.7×10^{-17}	4.8×10^{-17}

Table 3.

Comparing absolute errors of $u_{N,M}(x,t)$ *versus* the LWFOM [26] at $\sigma = 0.90$

	LWFOM [26]		Present Scheme $\lambda = 1$		Present Scheme $\lambda = 0.90$	
x	$t = 0.15$	$t = 0.25$	$t = 0.15$	$t = 0.25$	$t = 0.15$	$t = 0.25$
0.1	4.4×10^{-3}	7.4×10^{-2}	1.1×10^{-3}	3.9×10^{-4}	1.7×10^{-11}	4.4×10^{-11}
0.2	5.2×10^{-3}	7.7×10^{-2}	1.0×10^{-3}	3.9×10^{-4}	9.3×10^{-11}	2.3×10^{-10}
0.3	6.0×10^{-3}	7.8×10^{-2}	1.0×10^{-3}	3.8×10^{-4}	1.9×10^{-10}	4.9×10^{-10}
0.4	6.7×10^{-3}	8.0×10^{-2}	1.0×10^{-3}	3.6×10^{-4}	2.7×10^{-10}	6.9×10^{-10}
0.5	7.5×10^{-3}	8.0×10^{-2}	9.7×10^{-4}	3.5×10^{-4}	2.8×10^{-10}	7.1×10^{-10}
0.6	8.2×10^{-3}	8.0×10^{-2}	9.1×10^{-4}	3.2×10^{-4}	2.2×10^{-10}	5.5×10^{-10}
0.7	8.8×10^{-3}	7.9×10^{-2}	8.4×10^{-4}	3.0×10^{-4}	1.2×10^{-10}	3.1×10^{-10}
0.8	9.5×10^{-3}	7.7×10^{-2}	7.7×10^{-4}	2.7×10^{-4}	4.0×10^{-11}	1.0×10^{-10}
0.9	1.0×10^{-2}	7.4×10^{-2}	6.8×10^{-4}	2.4×10^{-4}	4.0×10^{-12}	1.0×10^{-11}
1.0	1.0×10^{-2}	7.1×10^{-2}	5.9×10^{-4}	2.1×10^{-4}	6.9×10^{-18}	1.7×10^{-17}

Table 2 compares the absolute errors of $u_{N,M}(x,t)$ with $\lambda = 1$ and with $\lambda = 0.75$ against those given by LWFOM [26] with $\sigma = 0.75$.

Table 3 compares the absolute errors of $u_{N,M}(x,t)$ with $\lambda = 1$ and with $\lambda =$

Table 4.

Comparing absolute errors of $u_{N,M}(x,t)$ *versus* the LWFOM [26] at $\sigma = 0.85$

	LWFOM [26]		Present Scheme $\lambda = 1$		Present Scheme $\lambda = 0.85$	
x	$t = 0.15$	$t = 0.25$	$t = 0.15$	$t = 0.25$	$t = 0.15$	$t = 0.25$
0.1	1.5×10^{-2}	3.8×10^{-2}	1.8×10^{-3}	6.6×10^{-4}	2.3×10^{-11}	5.4×10^{-11}
0.2	1.8×10^{-2}	4.4×10^{-2}	1.8×10^{-3}	6.5×10^{-4}	1.2×10^{-10}	2.9×10^{-10}
0.3	2.1×10^{-2}	5.0×10^{-2}	1.7×10^{-3}	6.3×10^{-4}	2.6×10^{-10}	6.2×10^{-10}
0.4	2.3×10^{-2}	5.5×10^{-2}	1.7×10^{-3}	6.1×10^{-4}	3.6×10^{-10}	8.6×10^{-10}
0.5	2.9×10^{-3}	3.8×10^{-2}	1.6×10^{-3}	5.8×10^{-4}	3.7×10^{-10}	8.9×10^{-10}
0.6	1.1×10^{-2}	5.0×10^{-2}	1.5×10^{-3}	5.4×10^{-4}	2.9×10^{-10}	6.9×10^{-10}
0.7	1.8×10^{-2}	6.2×10^{-2}	1.4×10^{-3}	5.0×10^{-4}	1.6×10^{-10}	3.8×10^{-10}
0.8	2.6×10^{-2}	7.3×10^{-2}	1.2×10^{-3}	4.6×10^{-4}	5.3×10^{-11}	1.2×10^{-10}
0.9	3.3×10^{-2}	8.3×10^{-2}	1.1×10^{-3}	4.1×10^{-4}	5.2×10^{-12}	1.2×10^{-11}
1.0	4.0×10^{-2}	9.3×10^{-2}	1.0×10^{-3}	3.5×10^{-4}	1.9×10^{-17}	0.00

0.90 against those given by LWFOM [26] with $\sigma = 0.90$.

Table 4 compares the absolute errors of $u_{N,M}(x,t)$ with $\lambda = 1$ and with $\lambda = 0.85$ against those given by LWFOM [26] with $\sigma = 0.85$.

The numerical results clearly demonstrate the superior accuracy of our spectral method across all test cases, particularly in handling the solution's singular behavior near the initial time.

5. CONCLUSION

In this study, we developed a fully spectral collocation method for solving the time-fractional S-KdV equation. The approach uses the strengths of spectral methods in both temporal and spatial directions by employing fractional-order Chebyshev functions and shifted Chebyshev polynomials as basis functions. The fractional Chebyshev basis is particularly well-suited for capturing the weakly singular behavior near the initial time, a challenge commonly encountered in fractional differential equations due to their nonlocal memory effects.

By transforming the original problem into a system of algebraic equations *via* the operational matrix technique, the proposed method achieves high-order accuracy while maintaining computational efficiency. Numerical experiments confirm the reliability and superiority of the method compared to existing techniques, especially in terms of resolving singularities and preserving fine-scale structures in the solution.

The present framework offers a promising direction for the numerical treatment of other classes of time-fractional evolution equations involving higher-order spatial derivatives.

Future research may explore extensions to multi-dimensional problems, adaptive basis selection strategies, and coupling with variable-order or tempered fractional derivatives to model more complex physical phenomena.

Acknowledgements. This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

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