A FRACTIONAL-ADAPTIVE GEGENBAUER SPECTRAL COLLOCATION METHOD FOR TIME-FRACTIONAL ALLEN-CAHN EQUATIONS

M.M. ALZUBAIDI[1], M.Z. YOUSSEF[2], F.A.H. ALOMARI[3], M.A. ZAKY[2], S.S. EZZ-ELDIEN[4,*]

[1]Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia
[2]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia
[3]Department of Mathematics, Faculty of Science, Al-Baha University, Al-Baha 65779, Saudi Arabia
[4]Department of Mathematics, Faculty of Science, New Valley University, El-Kharga, 72511, Egypt
[*]Corresponding author, Email: *s_sezeldien@yahoo.com*

July 28, 2025

The time-fractional Allen-Cahn equation with Caputo derivative order $\sigma \in (0,1]$ generalizes phase-field models to capture memory effects and anomalous diffusion in materials science. Its numerical solution faces the critical challenge of weak initial singularities that degrade the accuracy of conventional methods. To overcome this, we develop a spectral collocation scheme that combines fractional Gegenbauer functions in time and matches the solution behavior at $t = 0$ combined with shifted Gegenbauer polynomials in space. Our approach uses differentiation matrices for the fractional derivative computation and handles nonlinearity through collocation at Gegenbauer roots. Numerical results demonstrate spectral accuracy for both smooth solutions and challenging singular cases, validating our rigorous treatment of initial singularities.

Key words: Fractional-order Gegenbauer function, Allen-Cahn equation, Spectral collocation method, Differentiation matrix.

1. INTRODUCTION

Phase-field models, particularly the Allen-Cahn equation, are fundamental for simulating interfacial dynamics in materials science, fluid mechanics, and biology. The time-fractional Allen-Cahn equation generated by replacing the classical time derivative with a Caputo derivative of order $\sigma \in (0,1]$ expands this framework to capture memory effects and anomalous diffusion in complex systems. While this fractionalization enhances modeling capabilities for metastable and disordered states, it introduces severe weak initial singularities at $t = 0$ that degrade the accuracy of conventional numerical methods.

Many authors have been interested in investigating the time-fractional Allen-Cahn equations and providing efficient numerical solutions for them. Sohaib *et al.* [1] introduced a numerical scheme for solving time-fractional Allen-Cahn equation by approximating the time derivatives by Grunwald-Letnikov formula, whereas the finite difference scheme is used for spatial approximation. Liu *et al.* [2] applied L1 scheme to discretize the time fractional derivatives in the time-fractional Allen-Cahn equation and applied a stabilized approach and the scalar auxiliary variable technique to construct its numerical solutions. Hou *et al.* [3] applied a semi-implicit scheme based on the L1-Crank-Nicolson approach and the extended scalar auxiliary variable for the solution of time-fractional Allen-Cahn equations. Liu and Yang [4] introduced a fully discrete approach for the solution of time-fractional Allen-Cahn equation. In [4], the authors used the spectral Galerkin technique in the space direction and the Mittag-Leffler function combined with the piecewise polynomial interpolation for the time discretization. In [5], the authors applied the backward Euler convolution quadrature rule together with linear weighted stabilized, weighted convex splitting and convex splitting schemes for solving time-fractional AllenCahn equations. Recently, Zhang *et al.* [6] developed two fully discrete schemes based on the finite element approach and L1 scheme for the solution of time-fractional Allen-Cahn equation. We also refer to some recent works on fractional equations that are used to describe a series of phenomena occurring in diverse physical settings [7–12].

The numerical solution of time-fractional equations faces inherent difficulties due to the nonlocal nature of fractional operators and the weak initial singularities induced by Caputo derivatives [13, 14]. These singularities degrade the accuracy of conventional methods, as standard polynomial bases cannot efficiently capture the solution's asymptotic behavior near the origin. To overcome these constraints, we propose a singularity-adapted spectral collocation method that directly embeds the solution's intrinsic structure into the discretization basis [15]. The core innovation of our approach lies in the basis functions whose functional form intrinsically captures the t^{σ}-type singularity at $t = 0$. Unlike conventional polynomial bases used in existing methods, these fractional Gegenbauer functions embed the singularity directly through their expansion. The critical insight is that the terms $t^{n\lambda}$ in this expansion naturally form a minimal basis for approximating singular solutions. By selecting $\lambda = \sigma$ (the fractional order), the basis explicitly incorporates the solution's t^{σ} asymptotic behavior without requiring mesh grading or *a priori* singularity exponent knowledge. This functional design maintains spectral convergence for singular solutions.

The remainder of this paper is organized as follows. In Sec. 2, we present the fractional Gegenbauer functions and their key analytical properties, including orthogonality and explicit expansions that embed fractional-order singularities. Section 3 is devoted to the construction of a singularity-adapted spectral collocation scheme for solving the time-fractional Allen-Cahn equation. We derive the fully discrete formulation using fractional Gegenbauer functions in time and shifted Gegenbauer polynomials in space, along with explicit differentiation and fractional operators. In Sec. 4, we validate the proposed approach with numerical examples to demonstrate its accuracy and efficiency. Finally, concluding remarks and potential extensions are discussed in Sec. 5.

2. FRACTIONAL GEGENBAUER FUNCTION

We denote by $\mathcal{G}^{\alpha}_{L,j}(x),\ j \geq 0,\ \alpha > -\frac{1}{2},\ x \in \Lambda = [0,L]$, the shifted Gegenbauer polynomials, given analytically by:

$$\mathcal{G}^{\alpha}_{L,j}(x) = \sum_{k=0}^{j} \mathcal{E}^{\alpha}_{j,k} \frac{x^k}{L^k}; \qquad \mathcal{E}^{\alpha}_{j,k} = (-1)^{j-k} \frac{\Gamma(2\alpha+j+k)\Gamma(\alpha+\frac{1}{2})}{\Gamma(2\alpha)\Gamma(\alpha+k+\frac{1}{2})(j-k)!k!}.$$

These polynomials satisfy

$$\int_0^L \mathcal{G}^{\alpha}_{L,j}(x)\mathcal{G}^{\alpha}_{L,k}(x) w^{\alpha}_L(x) dx = \delta_{jk} h^{\alpha}_{L,k}, \tag{1}$$

and

$$w^{\alpha}_L(x) = (Lx - x^2)^{\alpha-\frac{1}{2}}, \quad h^{\alpha}_{L,k} = \frac{2^{1-4\alpha}\pi\Gamma(2\alpha+k)L^{2\alpha}}{(\alpha+k)k!\Gamma^2(\alpha)}. \tag{2}$$

Based on shifted Gegenbauer polynomials $\mathcal{G}^{\alpha}_{L,j}(x),\ j \geq 0$, we can expand any function $\Phi \in L^2_{w^{\alpha}_L}(\Lambda)$ by:

$$\Phi(x) = \sum_{j=0}^{\infty} \phi_j \mathcal{G}^{\alpha}_{L,j}(x); \qquad \phi_j = \frac{1}{h^{\alpha}_{L,j}} \int_0^L \Phi(x)\mathcal{G}^{\alpha}_{L,j}(x) w^{\alpha}_L(x) dx. \tag{3}$$

We define the orthogonal projection $\Pi^{\alpha}_{L,\gamma}$ as follows:

$$\begin{cases} \Pi^{\alpha}_{L,\gamma} : L^2_{w^{\alpha}_L}(\Lambda) \to \mathcal{G}^{\alpha}_{\gamma}; \qquad \mathcal{G}^{\alpha}_{\gamma} = \text{Span}\{\mathcal{G}^{\alpha}_{L,j}(x) : \qquad 0 \leq j \leq \gamma\}, \\ \Pi^{\alpha}_{L,\gamma}\Phi = \Phi_{\gamma}(x) = \sum_{j=0}^{\gamma} \phi_j \mathcal{G}^{\alpha}_{L,j}(x) = \Phi^T_{\gamma} \mathfrak{G}_{\alpha,\gamma}(x), \\ \Phi_{\gamma} = [\phi_j, \quad 0 \leq j \leq \gamma]^T, \qquad \mathfrak{G}_{\alpha,\gamma}(x) = [\mathcal{G}^{\alpha}_{L,j}(x), \quad 0 \leq j \leq \gamma]^T. \end{cases}$$

We denote by $\mathcal{FG}^{\alpha,\lambda}_i(t),\ i \geq 0,\ \alpha > -\frac{1}{2},\ t \in \mathcal{I} = [0,1],\ \lambda \in (0,1]$, the fractional-order Gegenbauer functions, given analytically by [16]:

$$\mathcal{FG}^{\alpha,\lambda}_i(t) = \mathcal{G}^{\alpha}_{1,j}(2t^{\lambda} - 1) = \sum_{n=0}^{i} \mathcal{E}^{\alpha}_{i,n} t^{n\lambda}, \tag{4}$$

and satisfy

$$\int_0^1 \mathcal{FG}^{\alpha,\lambda}_i(t)\mathcal{FG}^{\alpha,\lambda}_j(t)\, {}_f w^{\alpha,\lambda}(t) dt = \delta_{ij} h^{\alpha}_{1,i}; \quad {}_f w^{\alpha,\lambda}(t) = \lambda t^{\lambda(\alpha+\frac{1}{2})-1}(1-t^{\lambda})^{\alpha-\frac{1}{2}}.$$

The function $\Phi \in L^2_{{}_f w^{\alpha,\lambda}}(\mathcal{I})$ can be expanded by:

$$\Phi(t) = \sum_{i=0}^{\infty} \phi_i \mathcal{FG}^{\alpha,\lambda}_i(t); \qquad \phi_i = \frac{1}{h^{\alpha}_{1,i}} \int_0^1 {}_f w^{\alpha,\lambda}(t)\Phi(t)\mathcal{FG}^{\alpha,\lambda}_i(t) dt. \tag{5}$$

We define the span

$$\mathcal{F}_\rho = \text{Span}\{\mathcal{FG}_i^{\alpha,\lambda}(t): \qquad 0 \le i \le \rho\},$$

and the orthogonal projection ${}_f\Pi_\rho^{\alpha,\lambda} : L^2_{{}_fw^{\alpha,\lambda}}(\mathcal{I}) \to \mathfrak{F}_\rho^\alpha$, such that

$$\begin{cases} \left({}_f\Pi_\rho^{\alpha,\lambda}\Phi - \Phi, \pi\right)_{{}_fw^{\alpha,\lambda}} = 0, \qquad \forall \pi \in \mathfrak{FG}_{\rho,\lambda}^\alpha, \\ {}_f\Pi_\rho^{\alpha,\lambda}\Phi = \Phi_{\alpha,\rho} = \sum_{i=0}^{\rho} \phi_i \mathcal{FC}_i^{\alpha,\lambda}(t) = \Phi_{\alpha,\rho}^T \mathfrak{FG}_{\alpha,\rho,\lambda}(t), \\ \Phi_{\alpha,\rho} = [\phi_i, \quad 0 \le i \le \rho]^T, \qquad \mathfrak{FG}_{\alpha,\rho,\lambda}(t) = [\mathcal{FG}_i^{\alpha,\lambda}(t), \quad 0 \le i \le \rho]^T. \end{cases}$$

3. NUMERICAL SCHEME

The current Section is allocated to the numerical solution of the following time-fractional Allen-Cahn equation

$${}_CD_{0,t}^\sigma u(x,t) - \epsilon^2 \frac{\partial^2 u(x,y)}{\partial x^2} - u(x,t) + u^3(x,t) = \phi(x,t), \qquad x \in \Lambda,\ t \in \mathcal{I}, \quad (6)$$

subjected to

$$\begin{aligned} u(x,0) &= a(x), \qquad x \in \alpha, \\ u(0,t) &= b(t), \qquad u(L,t) = c(t), \qquad t \in \mathcal{I}, \end{aligned}$$

where $0 < \rho \le 1$, ϵ are known real constants and $\Phi(x,t)$, $a(x)$, $b(t)$, and $c(t)$ are known real functions.

The time-space spectral approach for (6) is to find $u_{\gamma,\rho} \in \mathcal{G}_\gamma \times \mathcal{FG}_\rho$, such that

$${}_CD_{0,t}^\sigma u_{\gamma,\rho}(x,t) - \epsilon^2 \frac{\partial^2 u_{\gamma,\rho}(x,t)}{\partial x^2} - u_{\gamma,\rho}(x,t) + u_{\gamma,\rho}^3(x,t) = \phi_{\gamma,\rho}(x,t), \quad (7)$$

where

$$\phi_{\gamma,\rho}(x,t) = \mathfrak{FG}_{\alpha,\rho,\lambda}^T(t)\Phi_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}(x), \quad (8)$$

with

$$\begin{aligned} \Phi_{\gamma,\rho} &= (\psi_{i,j})_{0\le i\le\rho,\ 0\le j\le\gamma}, \\ \psi_{i,j} &= \frac{1}{h_{L,j}^\alpha h_i^{\alpha,\lambda}} \int_0^1 \int_0^L w_L^\alpha(x)\, {}_fw^{\alpha,\lambda}(t)\phi(x,t)\mathcal{G}_{L,j}^\alpha(x)\mathcal{FG}_i^{\alpha,\lambda}(t)dxdt. \end{aligned}$$

If we denote

$$u_{\gamma,\rho}(x,t) = \mathfrak{FG}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}(x), \quad (9)$$

then we can write

$$\begin{aligned} {}_CD_{0,t}^\sigma u_{\gamma,\rho}(x,t) &= \left({}_CD_{0,t}^\sigma \mathfrak{FG}_{\alpha,\rho,\lambda}^T(t)\right)\mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x), \\ \frac{\partial^2}{\partial x^2} u_{\gamma,\rho}(x,t) &= \mathfrak{FG}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\left(\frac{\partial^2}{\partial x^2}\mathfrak{G}_{\alpha,\gamma}^L(x)\right). \end{aligned} \quad (10)$$

The next Theorems are highly useful in subsequent Sections.

Theorem 3.1 *[17] The derivative of order q for the vector $\mathfrak{G}_{\alpha,\gamma}(x)$ is given by:*

$$\frac{d^q}{dx^q}\mathfrak{G}_{\alpha,\gamma}(x) = \boldsymbol{D}^{(q)}\mathfrak{G}_{\alpha,\gamma}(x), \tag{11}$$

$$\boldsymbol{D}_q^\alpha = (d_{i,j}^{\alpha,q});\ d_{i,j}^{\alpha,q} = \begin{cases} C_q^\alpha(i,j), & i > j, \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$C_q^\alpha(i,j) = \frac{L^{2\alpha}(i+2\alpha+1)(i+2\alpha+2)_j(j+\alpha+2)_{i-j-1}\Gamma(j+2\alpha+1)}{(i-j-1)!\Gamma(2j+2\alpha+1)} \times {}_3F_2\left(\begin{matrix} j-i+1, & i+j+2\alpha+1, & j+\alpha+1 \\ j+\alpha+2 & 2j+2\alpha+2 & \end{matrix} ;1\right).$$

Theorem 3.2 *[18] The fractional derivatives of order σ, $0 < \sigma \le 1$, for the vector $\mathfrak{F}\mathfrak{G}_{\alpha,\mathcal{N},\lambda}(t)$ are given by:*

$${}_C D_{0,t}^\sigma \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}(t) = \Xi_{(\sigma)}\mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}(t), \tag{12}$$

where $\Xi_{(\sigma)} = \left(\xi_{j,i}^{\alpha,\sigma}\right)_{0\le j\le\rho,\ 0\le i\le\rho}$, and

$$\xi_\sigma^{\alpha,\lambda}(j,i) = \sum_{m=0}^{j}\sum_{n=0}^{i} \frac{\sqrt{\pi}\mathcal{E}_{j,m}^\alpha \mathcal{E}_{i,n}^\alpha \Gamma(m\lambda+1)\Gamma(m+n-\frac{\sigma}{\lambda}+\frac{1}{2})}{\lambda h_{1,i}^\alpha \Gamma(m\lambda-\sigma+1)\Gamma(m+n-\frac{\sigma}{\lambda}+1)}.$$

Combining the above two Theorems leads to

$$\begin{aligned} {}_C D_{0,t}^\sigma u_{\gamma,\rho}(x,t) &= \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\Xi_{(\sigma)}^T \mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x), \\ \frac{\partial^2}{\partial x^2}u_{\gamma,\rho}(x,t) &= \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\mathbf{D}^{(2)}\mathfrak{G}_{\alpha,\gamma}^L(x). \end{aligned} \tag{13}$$

Thanks to (9) and (13), we can write the residual $\mathcal{R}_{\gamma,\rho}(x,t)$ of (7) as follows:

$$\begin{aligned} \mathcal{R}_{\gamma,\rho}(x,t) = {}& \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\Xi_{(\sigma)}^T\mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x) - \epsilon^2\mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\mathbf{D}^{(2)}\mathfrak{G}_{\alpha,\gamma}^L(x) \\ & - \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x) + \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\mathcal{U}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x) \\ & - \mathfrak{F}\mathfrak{G}_{\alpha,\rho,\lambda}^T(t)\mathcal{G}_{\rho,\gamma}\mathfrak{G}_{\alpha,\gamma}^L(x). \end{aligned}$$

Finally, if we denote the roots of $\mathcal{G}_{\gamma+1}^L(x)$ by $x_{\gamma,\alpha}^L$ $(0 \le \alpha \le \gamma)$, and the roots of $\mathcal{F}\mathcal{G}_{\rho+1}^\lambda(t)$ by $t_{\rho,\beta}^{(\lambda)}$ $(0 \le \beta \le \rho)$, then the solution of the time-fractional Allen-Cahn equation (6) can be got by solving the following system:

$$\begin{aligned}
&\mathcal{R}_{\gamma,\rho}(x^L_{\gamma,\alpha},t^{(\lambda)}_{\rho,\beta})=0, && 1\le\alpha\le\gamma-1,\ 1\le\beta\le\rho,\\
&\mathfrak{F}\mathfrak{G}^T_{\alpha,\rho,\lambda}(0)\mathcal{U}_{\rho,\gamma}\mathfrak{G}^L_{\alpha,\gamma}(x^L_{\gamma,\alpha})=a(x^L_{\gamma,\alpha}), && 0\le\alpha\le\gamma,\\
&\mathfrak{F}\mathfrak{G}^T_{\alpha,\rho,\lambda}(t^{(\lambda)}_{\rho,\beta})\mathcal{U}_{\rho,\gamma}\mathfrak{G}^L_{\alpha,\gamma}(0)=b(t^{(\lambda)}_{\rho,\beta}), && 1\le\beta\le\rho,\\
&\mathfrak{F}\mathfrak{G}^T_{\alpha,\rho,\lambda}(t^{(\lambda)}_{\rho,\beta})\mathcal{U}_{\rho,\gamma}\mathfrak{G}^L_{\alpha,\gamma}(L)=c(t^{(\lambda)}_{\rho,\beta}), && 1\le\beta\le\rho.
\end{aligned}\tag{14}$$

4. NUMERICAL RESULTS

The current Section tests the accuracy of the proposed numerical scheme and confirm its superiority over other numerical schemes in the literature.

Example 1. Consider the following problem [19]:

$$\begin{aligned}
&{}_CD^{\sigma}_{0,t}u(x,t)-\frac{\partial^2u(x,t)}{\partial x^2}-u(x,t)+u^3(x,t)=\frac{2\sin(x)t^{2-\sigma}}{\Gamma(3-\sigma)}+t^6\sin^3(x),\\
&u(x,0)=0,\qquad 0\le x\le\pi,\\
&u(0,t)=u(\pi,t)=0,\qquad 0\le t\le 1,
\end{aligned}\tag{15}$$

where the exact solution is $u(x,t)=t^{\delta}\sin(x)$.

This example is postulated by Liu *et al.* [19] with $\delta=2$, who developed a finite difference approach with a Fourier spectral technique (FDFST) for approximating its solution. To confirm the superiority of the numerical scheme introduced in Section 3 over the FDFST [19], we apply our scheme to solve the current problem with $\alpha=1$. Table 1 compares the L_2-errors of $u_{\gamma,\rho}(x,t)$ with $\rho=10$ and $\gamma=\{4,8,12,16\}$ *versus* those given by the FDFST [19] at $\sigma=\lambda=0.2$. Table 2 compares the L_2-errors of $u_{\gamma,\rho}(x,t)$ with $\rho=4$ and $\gamma=\{4,8,12,16\}$ *versus* those given by the FDFST [19] at $\sigma=\lambda=0.5$. To test the proposed technique for a problem with a nonsmooth solution, Table 3 obtains the L_∞-errors of $u_{\gamma,\rho}(x,t)$ with $\delta=\sigma$, $\sigma=0.2$ and 0.5.

Table 1.

Comparing L_2-errors of $u_{\gamma,\rho}(x,t)$ *versus* the FDFST [19] at $\sigma=0.2$ for Example 1

FDFST [19]		FDFST [19]		Present Scheme	
M	$\Delta t=2^{-16}$	N	$h=2^{-12}$	γ	$\rho=10$
2^5	$4.3385.10^{-4}$	2^5	$3.4200.10^{-3}$	4	$3.6731.10^{-4}$
2^6	$1.1000.10^{-4}$	2^6	$2.0022.10^{-3}$	8	$7.0623.10^{-7}$
2^7	$2.9043.10^{-5}$	2^7	$1.0715.10^{-3}$	12	$1.9390.10^{-11}$
2^8	$8.8305.10^{-6}$	2^8	$5.5260.10^{-4}$	16	$1.8470.10^{-16}$

Table 2.

Comparing L_2-errors of $u_{\gamma,\rho}(x,t)$ *versus* the FDFST [19] at $\sigma = 0.5$ for Example 1

FDFST [19]		FDFST [19]		Present Scheme	
M	$\Delta t = 2^{-16}$	N	$h = 2^{-12}$	γ	$\rho = 4$
2^5	$3.8225.10^{-4}$	2^5	$2.0717.10^{-3}$	4	$2.5026.10^{-3}$
2^6	$9.5800.10^{-5}$	2^6	$1.0116.10^{-3}$	8	$4.8371.10^{-7}$
2^7	$2.4189.10^{-5}$	2^7	$4.8658.10^{-4}$	12	$1.3301.10^{-11}$
2^8	$6.3168.10^{-6}$	2^8	$2.3374.10^{-4}$	16	$1.2705.10^{-16}$

Table 3.

L_∞-errors of $u_{\gamma,\rho}(x,t)$ with $\delta = \sigma$ and $\sigma = 0.2,\, 0.5$ for Example 1

γ	$\sigma = 0.2,\ \rho = 10$	$\sigma = 0.5,\ \rho = 4$
4	$5.6063.10^{-3}$	$5.7213.10^{-3}$
8	$1.3330.10^{-6}$	$1.3423.10^{-6}$
12	$3.8996.10^{-11}$	$3.9135.10^{-11}$
16	$1.1102.10^{-15}$	$1.1379.10^{-15}$

Example 2. Consider the following problem [20]:

$$
\begin{aligned}
&{}_{C}D_{0,t}^{\frac{1}{2}}u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + u^3(x,t) = \phi(x,t),\\
&u(x,0) = 0, \qquad 0 \le x \le 1,\\
&u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1,
\end{aligned}
\tag{16}
$$

where $\phi(x,t)$ is chosen so that the exact solution is $u(x,t) = (t^2+1)\sin(2\pi x)$.

Table 4.

Comparing L_2-errors of $u_{\gamma,\rho}(x,t)$ *versus* the HIA [20] for Example 2

HIA [20]				Present Scheme	
N	$p = 1$	$p = 2$	$p = 3$	γ	$\rho = 4$
10	$2.23.10^{-4}$	$4.40.10^{-5}$	$3.54.10^{-5}$	4	$1.16.10^{-2}$
20	$8.44.10^{-5}$	$4.50.10^{-5}$	$8.92.10^{-5}$	8	$1.04.10^{-4}$
40	$4.52.10^{-5}$	$1.89.10^{-5}$	$4.42.10^{-4}$	12	$8.74.10^{-8}$
80	$2.95.10^{-5}$	$2.86.10^{-5}$	$1.06.10^{-4}$	16	$1.78.10^{-11}$

Wang *et al.* [20] considered the current problem and introduced a numerical solution for it. The authors in [20] applied a Hermite neural network technique based on Hermite interpolation approach (HIA) to approximate the fractional derivatives in the considered problem. Here, we applied the proposed numerical scheme with

$\alpha = 2$. Table 4 compares the L_2-errors of $u_{\gamma,\rho}(x,t)$ with $\rho = 4$ and $\gamma = \{4, 8, 12, 16\}$ *versus* those given by the HIA [20] at $\lambda = 0.5$.

5. CONCLUSION

In this work, we developed a singularity-adapted spectral collocation method for efficiently solving the time-fractional Allen-Cahn equation. By using the fractional Gegenbauer functions, the proposed method captures the intrinsic t^{σ}-type singularities present in the solution, ensuring high accuracy without requiring mesh refinement or prior knowledge of the singularity order. The use of spectral Gegenbauer polynomials in space, combined with the differentiation matrices for both integer and fractional derivatives, enables a fully discrete scheme with spectral convergence.

Numerical experiments confirm the method's effectiveness in resolving initial singular behavior and achieving superior accuracy compared to conventional approaches. This framework opens new avenues for solving a broad class of time-fractional partial differential equations exhibiting weak regularity, with potential extensions to higher dimensions and nonlinear systems.

Acknowledgements. This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

REFERENCES

1. M. Sohaib, A. Shah, K.M. Furati, H. Khaliq, A numerical scheme for time-fractional Allen-Cahn equation with application in phase separation, Inter. J. Comput. Math. **102**, 449-464 (2025).
2. Z. Liu, X. Li, J. Huang, Accurate and efficient algorithms with unconditional energy stability for the time fractional CahnHilliard and Allen-Cahn equations, Numer. Methods Partial Differ. Equ. **37**, 2613-2633 (2021).
3. D. Hou, H. Zhu, C. Xu, Highly efficient schemes for time-fractional Allen-Cahn equation using extended SAV approach, Numer. Algorithms **88** 1077-1108 (2021).
4. X. Liu, Y. Yang, Mittag-Leffler Interpolation Integrator for the Time-Fractional Allen-Cahn Equation, J. Sci. Comput. **104**, 15 (2025).
5. Q. Du, J. Yang, Z. Zhou, Time-Fractional AllenCahn Equations: Analysis and Numerical Methods, J. Sci. Comput. **85**, 42 (2020).
6. J. Zhang, X. Li, W. Ma, Two-Grid Finite Element Method for the Time-Fractional Allen-Cahn Equation With the Logarithmic Potential, Math. Meth. Appl. Sci., doi.org/10.1002/mma.10704 (2025).
7. B.A. Malomed, Basic fractional nonlinear-wave models and solitons, Chaos **34**, 022102 (2024).
8. W.A. Alrowaily, R. Shah, S. Alvaro, W. Alhejaili, C.G.L. Tiofack, S.M.E. Ismaeel, S.A. El-Tantawy, Analysis of fractional Swift-Hohenberg models using highly accurate techniques within the Caputo operator framework, Rom. Rep. Phys. **76**, 112 (2024).
9. H.O. Sidi, M. Babatin, M. Alosaimi, A.S. Hendy, M.A. Zaky, Simultaneous numerical inversion of space-dependent initial condition and source term in multi-order time-fractional diffusion

models, Rom. Rep. Phys. **76**, 104 (2024).
10. M.A. Zaky, A. Al Kenany, S. Alhazmi, S.S. Ezz-Eldien, Numerical treatment of tempered space-fractional Zeldovich-Frank-Kamenetskii equation, Rom. Rep. Phys. **77**, xxx (2025).
11. Guo-Qing Liu, Guo-Cheng Wu, Parallel computing and a multi-layer neural network algorithm for solving the fractional Duffing system, Rom. J. Phys. **69**, 107 (2024).
12. D. Mihalache, Localized structures in optical media and Bose-Einstein condensates: An overview of recent theoretical and experimental results, Rom. Rep. Phys. **76**, 402 (2024).
13. M.A. Saker, S.S. Ezz-Eldien, and A.H. Bhrawy, A pseudo spectral method for solving the time fractional generalized Hirota-Satsuma coupled Korteweg-de Vries system, Rom. J. Phys. **62**, 105 (2017).
14. M.A. Zaky, Finite difference/fractional Pertrov-Galerkin spectral method for the linear time-space fractional reaction-diffusion equation, Mathematics **13**, 1864 (2025).
15. S.S. Ezz-Eldien, W. Alharbi, M.M. Alzubaidi, M.A. Zaky, Fractional Chebyshev spectral scheme for time-fractional seventh-order KdV equations with non-smooth solutions, Rom. Rep. Phys., In press.
16. A.A. El-Kalaawy, E.H. Doha, S.S. Ezz-Eldien, M.A. Abdelkawy, R.M. Hafez, A.Z.M. Amin, D. Baleanu, and M.A. Zaky, A computationally efficient method of a class of fractional variational and optimal control problems using fractional Gegenbauer functions, Rom. Rep. Phys. **70**, 109 (2018).
17. E.H. Doha, A.H. Bhrawy, D. Baleanu, S.S. Ezz-Eldien, The Operational Matrix Formulation of the Jacobi Tau approximation for Space Fractional Diffusion Equation, Adv. Differ. Equ. **2014**, 231 (2014).
18. A.H. Bhrawy, M.A. Zaky, Shifted fractional-order Jacobi orthogonal functions: Application to a system of fractional differential equations, Appl. Math. Model. **40**, 832-845 (2016).
19. H. Liu, A. Cheng, H. Wang, J. Zhao, Time-fractional Allen-Cahn and Cahn-Hilliard phase-field models and their numerical investigation, Comput. Math. Appl. **76**, 1876-1892 (2018).
20. X. Wang, X. Wang, H. Qi, H. Xu, Numerical simulation of time fractional Allen-Cahn equation based on Hermite neural solver, Appl. Math. Comput. **491**, 129234 (2025).