SHAPES OF ROGUE WAVE PERTURBATIONS FOR PHYSICAL EQUATIONS

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We consider perturbations on the nonlinear Schrödinger equation and the Boussinesq equation. These involve dispersion terms of various orders, higher-order nonlinearities, and self-steepening terms. For example, in some cases these can combine to constitute the Hirota or Sasa-Satsuma equations, but mostly they result in non-integrable equations, so there is no simple exact solution. We determine the actual shape changes induced by the imposition of perturbations and plot these.

1. MOTIVATION

In the strict sense of the word, solitons are pulses that can pass through each other without change in shape, and this only occurs when the physics is described by an integrable system like the nonlinear-Schrödinger equation (NLSE), Korteweg-de Vries (KdV) equation, modified Korteweg-de Vries (mKdV) equation, etc.

Over the years, ‘soliton’ has been used more loosely to describe any pulse that propagates forever without change of shape, even though it may not satisfy the lossless collision condition described above. Hence the name is retained in the class of ‘dissipative solitons’ [1].

Stories of huge oceanic rogue waves sinking ships have fascinated people for centuries. The interest increased after the first scientific measurement of one, $viz.$ the ‘Draupner wave’ in 1995. Some physical explanation of the growth and decay of such a wave was given in [2, 3]. From 2009 onwards, higher-order rogue waves solutions have been found [4], and these have been verified by observations in water tank experiments [5]. Multi-component rogue waves have also been presented. For example, a non-recursive Darboux transformation formalism was used to obtain a hierarchy of rogue wave solutions to the Manakov system in [6].

There have been some excellent review papers covering this research theme. Generating mechanisms are covered in [7], while multidimensional nonlinear systems are treated in [8]. Many topics, including the Peregrine wave, higher-order rational solutions of the NLSE, and classification of the rogue wave patterns, including circular structures that resemble atoms with electron shells, are summarized in [9]. Then, [10] covers rogue waves in nonlinear optics. Further structures in optical media are discussed, as well as the application to Bose-Einstein condensates, in [11].

A few authors have considered the question as to whether rogue waves can exist in non-integrable systems. A perturbation solution for deviations from the NLSE was introduced in [12]. More recently, the approach of that paper has been verified in [13] and further numerical calculations have been made. The term 'rogue wave' (RW) is used in papers like [14, 15] to describe some cases of evolution from a starting condition in a non-integrable system, but then, no isolated rogue wave is seen, and it seems unreasonable to use the term for these cases, since there is just an irregular pattern of 'hills and valleys'. So it is still an open question as to whether any non-integrable system can support exact rogue waves. It appears more likely that RWs, in the strict sense, occur as a type of 'resonance' that depends on the system being integrable.

However, for perturbations that are small, as considered in this paper, the resultant formation may just look like a modified regular rogue wave, and this can appear and disappear before the perturbation has much long-term effect, *e.g.* by causing modulation instability. Deviations from the Sasa-Satsuma and Hirota equations have been considered in [16]. These are further extended here in Sec. 3. The purpose of this paper is to find and illustrate the shapes of perturbing functions caused by various terms added to the original equation. Each term has some physical significance, *e.g.* dispersion terms of various orders, self-steepening terms, and 'quintic' and other higher-order nonlinearities.

2. NONLINEAR SCHRÖDINGER EQUATION (NLSE

We take the NLSE [17] with a perturbing term for higher order dispersion:

$$iu_x(x,t)+\frac{1}{2}u_{tt}(x,t)+u(x,t)^2u^*(x,t)+\epsilon S=0. \tag{1}$$

and let $u=pe^{ix}, S=Re^{ix}$. The perturbing term depends on various amplitudes and derivatives, *e.g.* $R=R(u_t,u_{tt},u_{ttt},u,u^*,|u|)$. We let $p=p_0+\epsilon r(x,t)$. So when $\epsilon=0$, then the basic solution of the NLSE is $u_0=p_0\,e^{ix}$ where

$$p_0=4\frac{1+2ix}{1+4x^2+4t^2}-1.$$

Throughout this paper, we use $D=D(x,t)=1+4x^2+4t^2$, so here $p_0=\frac{4}{D}(1+2ix)-1$. Then

$$\epsilon R(x,t)+\frac{1}{2}p_{tt}+ip_x(x,t)+p(x,t)(pp^*-1)=0.$$

So

$$D^2[\frac{1}{2}\left(r_{tt}+2ir_x\right)+R(x,t)]+\left[16t^4+8t^2\left(4x^2-7\right)+16x^4+72x^2+17\right]r(x,t)$$

$$+\left[4t^2+4x(x-2i)-3\right]^2 r^*(x,t)=0.$$

For the initial Sections, the perturbing term is $S = S_n = d_n(i)^n u^{(0,n)}(x,t)$, where d_n is a coefficient marking the nth order dispersion.

Hence, $R = R_n = d_n(i)^n p^{(0,n)}(x,t)$. We get:

$$i^n d_n \epsilon p_{n,t} + \frac{1}{2}p_{tt}(x,t) + ip_x + p(x,t)^2 p^*(x,t) - p(x,t) = 0, \tag{2}$$

where $p_{n,t}$ is the n^{th} derivative of p with respect to t.

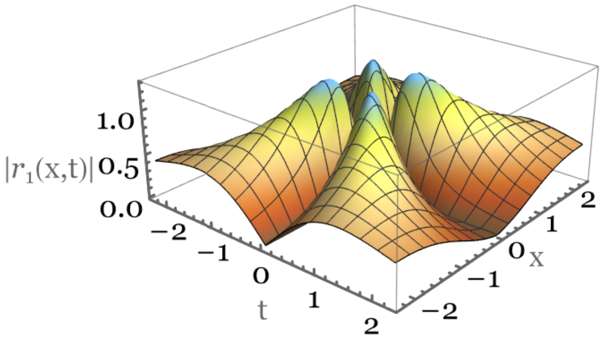

Fig. 1 – *First order* $|r_1|$ *perturbing function, where* $r_1 = \frac{32t(1+2ix)x}{D^2(x,t)}$*. So the full solution is given by* $p = p_0 + \epsilon d_1 r_1(x,t)$*, where* ϵ *is small. So,* $|r_1| = \frac{32|tx|\sqrt{4x^2+1}}{(4t^2+4x^2+1)^2}$*. There are four maxima; these occur when* $x = \pm\frac{1}{2}, t = \pm\frac{1}{\sqrt{6}}$*. In each case, the max is* $\frac{3\sqrt{3}}{4} \approx 1.3$*, so we need to ensure that* $1.3\epsilon d_1 \ll 1$*.*

In fact, we mostly set $d_n = 1$. The volume of a rogue wave is [18, 19]:

$$V = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|u(x,t)|^2-1)^2 dx\, dt. \tag{3}$$

So the volume can deviate from unity by a term of order ϵ.

2.1. NONLINEAR-SCHRÖDINGER EQUATION, n=1

So, if $d_1 = 1$ we have the $n = 1$ case:

$$4\left(t^2+x^2\right)\left(2t^2+2x^2+1\right)\left(r_{tt}+2ir_x\right)$$

$$+\left[4t^2+4x(x-2i)-3\right]^2 r^*(x,t) + 32d_1 t(2x-i)$$

$$+\frac{1}{2}r^{(0,2)}(x,t) + ir^{(1,0)}(x,t)$$

$$+\left[16t^4+8t^2\left(4x^2-7\right)+16x^4+72x^2+17\right] r = 0.$$

Now r takes the form of the following trial function: $r = r_1 = \frac{(1+2ix)q(x,t)}{(4t^2+4x^2+1)^2}$, with $q(x,t)$ real. We find that the solution is

$$q_x = 32t,$$

So $q = 32xt + f_1(t)$, with $f_1(t)$ having powers of t not greater than 1. Hence $f_1(t) = c_1 t$, with c_1 being a constant. Setting this to zero, we find, for $d_1 = 1$:

$$q = 32tx.$$

Now r_1 is plotted in Fig. 1. This gives $V_1 = 1 + \frac{4d_1^2\epsilon^2}{3}$, so $V_1 = 1$ to order ϵ.

2.2. NONLINEAR SCHRÖDINGER EQUATION, $n = 2$: SECOND ORDER DISPERSION

Now with $d_2 = 1$ we have the $n = 2$ case: With $r = r_2 = \frac{(1+2ix)q_2(x,t)}{(4t^2+4x^2+1)^2}$, we find:

$$q_{2,x} = 0$$

so $q_2 = f_2(t)$. The second equation then shows $f_2(t) = -32t^2 + c_2 t$. Setting the constant $c_2 = 0$ gives the $n = 2$ solution:

$$q_2 = -32t^2 d_2.$$

So

$$r = r_2 = -\frac{32t^2\,(1+2ix)}{(4t^2+4x^2+1)^2}.$$

We plot the second order $|r_2|$ perturbing function in Fig. 2.

In this case, the dispersion is of the same form as the dispersion term in the original NLSE. Hence, we can write down the exact solution:

$$p(x,t) = \frac{4(1+2ix)}{\frac{4t^2}{1-2d_2\epsilon} + 4x^2 + 1} - 1.$$

This shows that $V_2 = \sqrt{1-2d_2\epsilon} \approx 1 - d_2\epsilon$. To order ϵ, this gives:

$$p(x,t) \approx -1 + \frac{4(1+2ix)}{4t^2+4x^2+1} - \frac{32d_2 t^2(1+2ix)\epsilon}{(4t^2+4x^2+1)^2} + \cdots.$$

So this verifies that $q_2 = -32t^2 d_2$ as above, and shows that $V_2 = 1 - d_2\epsilon$ to order ϵ, agreeing with the above.

2.3. NLSE: THIRD ORDER DISPERSION

This time, $r = r_3 = \frac{q_3(x,t)}{(4t^2+4x^2+1)^2}$. We find that the solution is

$$q_3 = 48id_3t\left[4t^2 + (-2x+i)^2\right]. \tag{4}$$

This agrees with the answer found in [12]. We plot this third order perturbing function $|r_3|$ (with $d_3 = 1$) in Fig. 3. The maximum occurs on two circular 'craters'.

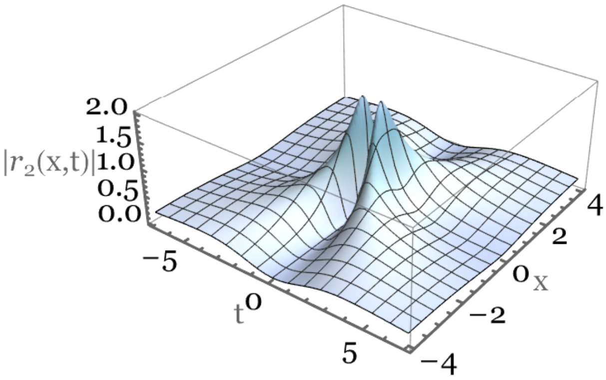

Fig. 2 – *Second order* $|r_2|$ *perturbing function, where* $r_2 = -\frac{32t^2(1+2ix)}{D(x,t)^2}$. *So the full solution is given by* $p = p_0 + \epsilon d_2 r_2(x,t)$, *where* ϵ *is small. When x=0, the function is real,* $r_2 = -\frac{32t^2}{(4t^2+1)^2}$ *and* $|r_2|$ *reaches its maximum when* $t = \pm\frac{1}{2}$. *In each case, the maximum is 2, so we need to ensure that* $2\epsilon << 1$.

3. PERTURBATIONS ON THIRD ORDER EQUATIONS: GENERAL

We can include other perturbations (*e.g.* see [16]):

$$iu_x + \frac{1}{2}u_{tt} + u(x,t)|u|^2 + i\epsilon[a_f u(x,t)(|u|^2)_t - d_3\, u_{ttt} + s_a(|u|^2\, u)_t] = 0. \tag{5}$$

Here, the coefficients s_a, a_f and d_3 are three independent parameters (of order 1). So we have included a self-steepening term, $s_a\,\frac{\partial}{\partial t}(|u|^2 u)$, a term related to the self-frequency shift, $a_f u\,\frac{\partial}{\partial t}(|u|^2)$, and a 3^{rd} order dispersion term, $d_3 u_{ttt}$ [12,20].

The overall factor ϵ must be small, indicating that the three terms are perturbations. When $\epsilon = 0$, the equation reduces to the NLSE. Now $r = r_3 = \frac{q_3(x,t)}{D^2(x,t)}$, as above. The form of the perturbation term needed in the solution is :

$$q_3 = \epsilon\, d_3\, 8t[4xk(x,t) + if(x,t)], \tag{6}$$

where $k(x,t)$ turns out to be a constant, say k_3. The real part shows that

$$f(x,t) = 16t^2(a_f + 3d_3) + k_3\left(-4t^2 + 4x^2 - 1\right)$$

$$+ s_a\left(28t^2 + 4x^2 + 1\right) + y_7(t)\exp[-\frac{8}{4t^2 + 4x^2 + 1}]$$

$$-8e^{-\frac{8}{4t^2+4x^2+1}}\, Ei\left(\frac{8}{4t^2 + 4x^2 + 1}\right)(3a_f + 6d_3 - k_3 + 5s_a)$$

where Ei is the exponential integral function. Thus:

$$f(x,t) = -g_3(x,t)(3a_f + 6d_3 - k_3 + 5s_a) + 16t^2(a_f + 3d_3)$$

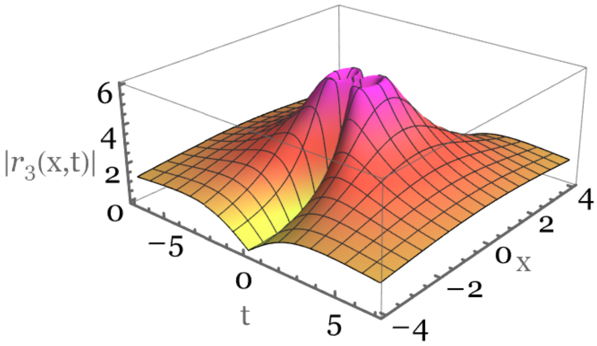

Fig. 3 – *Third order* $|r_3|$ *perturbing function. So the full solution is given by* $p = p_0 + \epsilon d_3 r_3(x,t)$*, where* ϵ *is small. When x=0, the function reaches its maximum when* $t = \pm\frac{1}{2}\left(1+\sqrt{2}\right) \approx \pm 1.20711$ *and when* $t = \pm\frac{1}{2}(\sqrt{2}-1) \approx \pm 0.20711$*. In each case, and on the two circular 'craters' seen, i.e.* $(t \pm \frac{1}{\sqrt{2}})^2 + x^2 = \frac{1}{4}$ *, the maximum is 6, so we need to ensure that* $6|\epsilon|d_3 \ll 1$*. The minimum is zero and this occurs for* $t = 0$ *and for* $(x,t) = (0, \pm 1/2)$.

$$+k_3\left(-4t^2+4x^2-1\right)+s_a\left(28t^2+4x^2+1\right)+y_7(t)\exp[-\frac{8}{4t^2+4x^2+1}],$$

where the function $g_3(x,t)$ is unbounded, so its coefficient must be zero, *i.e.*

$$k(x,t) = k_3 = 3a_f + 6d_3 + 5s_a.$$

Then

$$f(x,t) = \left(-4t^2+4x^2-1\right)(3a_f+6d_3+5s_a)+16t^2(a_f+3d_3)$$
$$+s_a\left(28t^2+4x^2+1\right)+y_7(t)\exp[-\frac{8}{4t^2+4x^2+1}].$$

Solving with this shows that $y_7(t) = 0$ and so

$$f(x,t) = 4t^2(a_f+6d_3+2s_a)+12x^2[a_f+2(d_3+s_a)]-3a_f-6d_3-4s_a.$$

If $a_f = s_a = 0$, then this reduces to the solution of Eq. 4. We can find the volume for this case:

$$V_a = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|u(x,t)|^2-1)^2 dx\,dt = 1$$
$$+\epsilon^2\left[s_a(11a_f+24d_3)+3(a_f+2d_3)^2+11s_a^2\right]. \tag{7}$$

So, to order ϵ, the volume retains its value of unity.

3.1. PERTURBATIONS ON THIRD ORDER EQUATIONS: HIROTA EQUATION

If we set the small parameter $\epsilon = \alpha, d_3 = 1, a_f = 6, s_a = -6$, then we obtain the Hirota equation:

$$iu_x + \frac{1}{2}u_{tt} + u(x,t)|u|^2 - i\epsilon(u_{ttt} + 6u_t|u|^2) = 0. \tag{8}$$

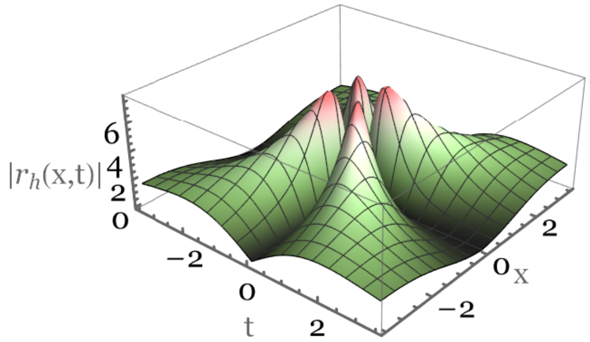

Fig. 4 – *Third order* $|r_h| = \frac{192}{D^2(x,t)}|tx(1+2ix)|$ *perturbing function, equation (9) from equation (8) for Hirota equation (*$d_3 = 1, a_f = 6, s_a = -6$ *in Section 3), i.e. equation 10. So the full solution is given by* $p = p_0 + \epsilon r_h(x,t)$*, where* ϵ *is small. The maximum is about 8, so we need to ensure that* $8\epsilon \ll 1$*. This has a similar form to the second order perturbing function of Fig. 1.*

The volume $V_h = 1$, correct to order α. Now, using Eq. (6), its solution is $u = pe^{ix}$, where:

$$p = p_0 + \epsilon r_h,$$

$p_0 = 4\frac{1+2ix}{D(x,t)} - 1$ and

$$r_h = -\frac{192tx}{D^2(x,t)}(1+2ix). \tag{9}$$

So

$$ue^{-ix} \approx -1 + 4\frac{1+2ix}{D(x,t)}[1 - 48\epsilon\frac{tx}{D(x,t)}]. \tag{10}$$

Now we can compare this with the exact solution for the 'NLSE+Hirota' equation [21], *i.e.*

$$ue^{-ix} = 4\frac{1+2ix}{1+4x^2+4(t+6\alpha x)^2} - 1. \tag{11}$$

This shows that $V_h = 1$ exactly, for any α.

Expanding to order α, we obtain:

$$ue^{-ix} \approx -1 + 4\frac{1+2ix}{D(x,t)}[1 - 48\alpha\frac{tx}{D(x,t)}].$$

This is the same (since $\epsilon = \alpha$) as Eq. (10), thus verifying the perturbation result.

We plot the magnitude of the perturbation term, $|r_h| = |\frac{192tx}{D^2(x,t)}(1+2ix)|$ in Fig. 4.

3.2. PERTURBATIONS AROUND THE SASA-SATSUMA EQUATION

To study perturbations of the Sasa-Satsuma equation (SSE), we employ coefficients s_a, a_f and d_3, so that the full expression is still Eq. (5).

For the Sasa-Satsuma equation itself, we have $s_a = -6, a_f = 3, d_3 = 1$.

We can approximate the solution around the integrable SSE case by taking s_a being arbitrary and considering the line $a_f = -3 - s_a$. Using the parameters satisfying this relation [16] allows us to get approximate solutions. These cases are not integrable unless $s_a = -6$. To first order in ϵ, the solution is:

$$\frac{u(x,t)}{e^{ix}} = p = p_0 + \epsilon\, r_{SSE}(x,t) = -1 + \frac{4}{D}(1+2ix) + \frac{8t\,\epsilon}{D^2}[4k_n x + i f_a(x,t)], \quad (12)$$

where $D = 1 + 4x^2 + 4t^2$. We now find that $k_n = 2s_a - 3$ and the polynomial $f_a(x,t)$ is:

$$f_a(x,t) = 3 - s_a + 4t^2(s_a+3) + 12x^2(s_a - 1). \qquad .$$

We plot the magnitude of $r_{SSE} = \frac{8t}{D^2}[4k_n x + i f_a(x,t)]$ for one example in Fig. 5.

To order ϵ, the volume retains its value of unity.

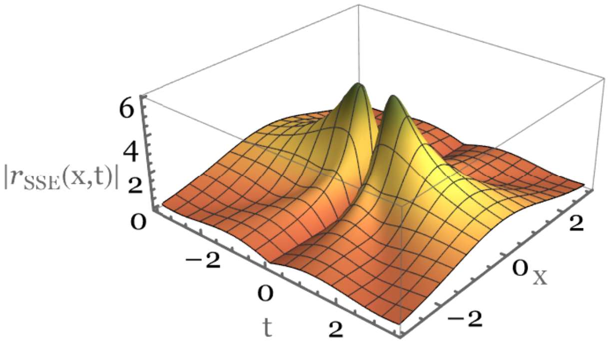

Fig. 5 – *Third order* $|r_{SSE}|$ *perturbing function. So the full solution is given by* $p = p_0 + \epsilon r_{SSE}(x,t)$, *where* ϵ *is small. Here* $s_a = 0, a_f = -3, d_3 = 1$. *Hence* $k_n = -3, f_a = 3(1+4t^2-4x^2)$, *so* $r_{SSE} = \frac{24it}{D^2}\left(4t^2 - 4x^2 + 4ix + 1\right)$. *The maximum magnitude is 6, occurring at* $(x,t) = (0, \pm\frac{1}{2})$, *so we need to ensure that* $6\epsilon \ll 1$.

4. FOURTH ORDER ($n = 4$): PERTURBATIONS ON NONLINEAR SCHRÖDINGER EQUATION

4.1. FOURTH ORDER DISPERSION AND A QUINTIC TERM

We now change the term perturbing the nonlinear Schrödinger equation to include a fourth order dispersion and a quintic term, first considered in [22] :

$$R = u_0(x,t)\,|u_0(x,t)|^4 - \frac{1}{6}(u_0)_{tttt}.$$

Then $r_4 = \frac{1}{2} + \frac{2}{D^3}[q_{4r} + 2ix\, q_{4i}]$, where

$$q_{4r} = -16\left(7t^4 + t^2\left(24x^2 - 5\right) + 17x^4\right) - 88x^2 - 5,$$
$$q_{4i} = -80t^4 - 64t^2\left(3x^2 - 2\right) - 8x^2\left(14x^2 + 1\right) + 5, \tag{13}$$

i.e.

$$r_4 = \frac{256}{D^3}t^2(1+2ix) - \frac{2}{D}(17+14ix) + \frac{8}{D^2}\left[8t^2 + \left(2t^2+3\right)(1+2ix)\right] + \frac{1}{2}, \tag{14}$$

where $D = 4t^2 + 4x^2 + 1$, as before.

Then $|r_4|$ has a maximum of ≈ 11.64 (when $t = 0, x \approx \pm 0.301$). It is plotted in Fig. 6. It has zeros on the x axis when $t = \pm 0.249, \pm 0.954, \pm 2.297$ and has a background value of 1/2.

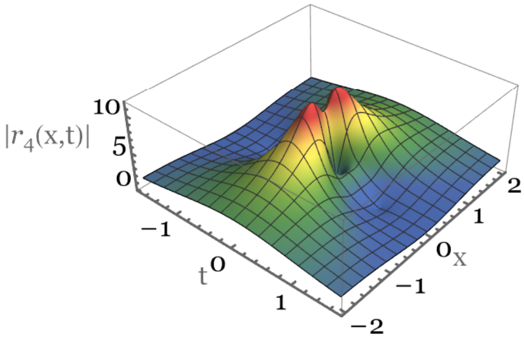

Fig. 6 – *Fourth order $|r_4|$ perturbing function, using equation (14). So the full solution is given by $p = p_0 + \epsilon r_4(x,t)$, where ϵ is small. The maximum is about 11.64, so we need to ensure that* $11.64\epsilon \ll 1$.

The background intensity level ($d_4 = 1$) is plainly $|u_b| = |-1 + \frac{\epsilon}{2}|$. Here we then have:

$$V_4 = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|u|^2 - |u_b|^2)^2\, dx\, dt \approx 1 - \frac{53\epsilon}{6}.$$

4.2. COMBINED DISPERSION, HIGH-ORDER NONLINEARITY, AND OTHER TERMS

We now change the perturbing factor to include more terms added to 4^{th} order dispersion :

$$R = s_4(u_0)_{tttt} + u_0|u_0|^4 + \frac{1}{108}(47 + 576s_4)(u_0)_{tt}|u_0|^2$$
$$+\frac{1}{108}(576s_4 - 37)u_0|u_{0t}|^2 + (6s_4 - \frac{1}{3})u_0^* u_{0t}^2 + \frac{17}{36}u_0^2(u_0^*)_{tt},$$

with s_4 being an arbitrary parameter.

Then $u = pe^{ix}$ where $p = p_0 + \epsilon r_{4z}$. Then $r_{4z} = \frac{1}{2} + \frac{2}{D^3}[q_{4v}(x,t) + 2ixq_{4w}(x,t)]$, where

$$q_{4v}(x,t) = \frac{128}{27}(72s_4 - 5)t^2\left(t^2 + x^2\right),$$

and

$$q_{4w}(x,t) = \frac{4}{27}\{8(288s_4 + 7)t^4 + 2t^2\left[4(288s_4 - 11)x^2 + 9\right] - 9\left(4x^2 + 1\right)^2\}, \quad (15)$$

with s_4 being an arbitrary parameter. An example is plotted in Fig. 7.

Here, again we have $|u_b| = |-1 + \frac{\epsilon}{2}|$, and

$$V_{4z} \approx 1 + \frac{\epsilon}{3}\left(8s_4 - \frac{37}{18}\right).$$

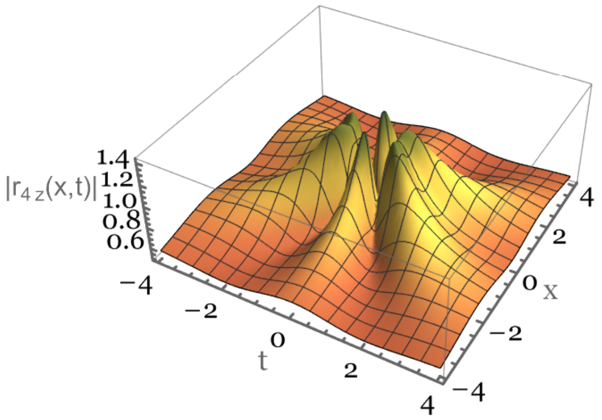

Fig. 7 – *Fourth order $|r_{4z}|$ perturbing function, using equation (15). So the full solution is given by $p = p_0 + \epsilon r_{4z}(x,t)$, where ϵ is small. Here $s_4 = 0.2$. The maximum is about 1.4, so we need to ensure that $1.4\epsilon \ll 1$.*

5. HIGHER ORDER NONLINEAR-SCHRÖDINGER EQUATIONS REGARDED AS PERTURBATIONS ON THE BASIC ONE

5.1. FOURTH ORDER ($n = 4$) NLSE ROGUE WAVE

We now consider the combined second and fourth order NLSE. The fourth order part can be viewed as a perturbation on the basic second order NLSE.

We now add additional terms, and actually study the fourth order ($n = 4$)

nonlinear-Schrödinger equation regarded as perturbation. We take

$$iu_x + \sum_{j=1}^{\infty} \alpha_{2j} K_{2j}[u] = 0, \tag{16}$$

where, in this subsection, $\alpha_2 = 1/2, \alpha_4 = \epsilon$ and the other $\alpha' s$ are zero. The functionals are [21]:

$$K_2[u(x,t)] \quad = \quad u_{tt} + 2u|u|^2$$

and

$$K_4[u(x,t)] = u_{tttt} + 8|u|^2 u_{tt} + 6u|u|^4 + 4u|u_t|^2 + 6u_t^2 u^* + 2u^2 u_{tt}^*.$$

The exact rogue wave solution (for any ϵ) is [21]

$$u(x,t) = \left[4\frac{1+2i(12\epsilon+1)x}{4(12\epsilon+1)^2x^2+4t^2+1} - 1 \right] e^{i(6\epsilon+1)x} \tag{17}$$

We set $u = pe^{i(1+6\epsilon)x}$ and $u_0 = p_0 e^{ix}$. We now restrict ourselves to small $|\epsilon|$. So, to order ϵ, $p - p_0 = \epsilon r_4$, where

$$r_4 = \frac{96i}{D^2} x \left(4t^2 - 4x^2 + 4ix + 1\right), \tag{18}$$

and $D = 4t^2 + 4x^2 + 1$. We plot $|r_4|$ from this in Fig. 8.

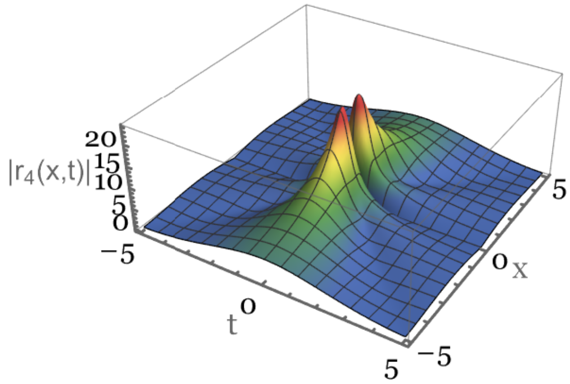

Fig. 8 – *Fourth order* $|r_4|$ *from equation (18). Here, the fourth order (n=4) nonlinear-Schrödinger equation is regarded as perturbation on the basic NLSE. So the full solution is given by* $p = p_0 + \epsilon r_4(x,t)$, *where* ϵ *is small. The maximum is 24, occurring at* $t = 0, x = \pm\frac{1}{2}$, *so we need to ensure that* $24\epsilon \ll 1$.

We find

$$V_4 = \frac{1}{1+12\epsilon} \tag{19}$$

5.2. SIXTH ORDER ($n = 6$) NLSE ROGUE WAVE

Furthermore, we now consider the combined second and sixth order NLSE. The sixth order part can be viewed as a perturbation on the basic second order NLSE. To study this, we now take $\alpha_2 = 1/2, \alpha_4 = 0, \alpha_6 = \epsilon$ and all other $\alpha's$ being 0 in eq. 16, where the new functional is K_6.

Further, taking $j = 3$, we find K_6 as the sixth order, *i.e.* sextic operator. It includes a sixth order derivative [21] and can be written:

$$K_6[\psi] = \psi_{tttttt} + \left[60\psi^*|\psi_t|^2 + 50(\psi^*)^2\psi_{tt} + 2\psi^*_{tttt}\right]\psi^2$$
$$+\psi\left[12\psi^*\psi_{tttt} + 8\psi_t\psi^*_{ttt} + 22|\psi_{tt}|^2\right] + \psi\left[18\psi_{ttt}\psi^*_t + 70(\psi^*)^2\psi_t^2\right] + 20(\psi_t)^2\psi^*_{tt}$$
$$+10\psi_t\left[5\psi_{tt}\psi^*_t + 3\psi^*\psi_{ttt}\right] + 20\psi^*\psi_{tt}^2 + 10\psi^3\left[(\psi^*_t)^2 + 2\psi^*\psi^*_{tt}\right] + 20\psi|\psi|^6. \quad (20)$$

The exact rogue wave solution (for any ϵ) is [21]

$$u(x,t) = \left[4\frac{1+2iB_3x}{4B_3^2x^2+4t^2+1} - 1\right]e^{i\phi_3 x} \quad (21)$$

where $B_3 = 1+60\epsilon$ and $\phi_3 = 1+20\epsilon$. We set $u = pe^{i\phi_3 x}$ and $u_0 = p_0e^{ix}$. We now restrict ourselves to small $|\epsilon|$. So, to order ϵ, $p - p_0 = \epsilon r_6$. We have

$$r_6 = \frac{480ix}{D^2}\left(4t^2 - 4x^2 + 4ix + 1\right). \quad (22)$$

We find

$$V_6 = \frac{1}{B_3} = \frac{1}{1+60\epsilon} \quad (23)$$

5.3. GENERAL HIGH-ORDER (ANY j) NLSE ROGUE WAVE

In fact, if we set $\alpha_2 = 1/2$, and then only include one higher-order functional, $K_{2j}, j > 1$, with $\alpha_{2j} = \epsilon$ and all other $\alpha's$ being 0, then

$$u(x,t) = \left[4\frac{1+2iB_jx}{4B_j^2x^2+4t^2+1} - 1\right]e^{i\phi_j x}, \quad (24)$$

where $B_j = 1+F_j\epsilon$ [23], $\phi_j = 1+\frac{(2j)!}{(j!)^2}\epsilon$, and $F_j = \frac{j(2j)!}{(j!)^2}$. This reduces to Eq. (17) when $j = 2$ and to Eq. (21) when $j = 3$. To order ϵ, we have $p - p_0 = \epsilon r_{2j}$. So

$$r_{2j} = \frac{8iF_jx}{D^2}\left(4t^2 - 4x^2 + 4ix + 1\right) = \frac{8ijx(2j)!}{(j!)^2D^2}\left(4t^2 - 4x^2 + 4ix + 1\right),$$

where $D = 4t^2 + 4x^2 + 1$. So the shape of $|r_{2j}|$ is the same as in Fig. 8, and only the vertical scale changes with j. Its maximum value is $2F_j$, and this occurs at $t = 0, x = \pm\frac{1}{2}$; so it is 24 when $j = 2$, as Fig. 8 shows, while it is 120 when $j = 3$ and 560 when $j = 4$, etc.

We find

$$V_{2j} = \frac{1}{B_j} = \frac{1}{1+\epsilon F_j}. \tag{25}$$

In fact, Eqs. (19), (23), and (25) use exact results for the full equation, so we do not require ϵ to be small.

6. PERTURBATION ON BOUSSINESQ EQUATION

Now we discuss the perturbation terms associated with rogue wave solutions $u = u(x,t)$ of the perturbed Boussinesq [24] equation :

$$u_{tt} + u_{xx} - (u^2)_{xx} - \frac{1}{3}u_{xxxx} + \epsilon\, S(x,t) = 0, \tag{26}$$

where subscripts denote partial derivatives, and the perturbation equation term, $S(x,t)$, can involve various derivatives. Equation (26) (with $S(x,t) = 0$) was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water [25,26].

The first order rogue wave solution is [27]: $u = u_0(x,t) = 2\frac{\partial^2}{\partial x^2}\ln[F(x,t)]$, where $F(x,t) = F_0(x,t) = 1 + x^2 + t^2$, so

$$u = u_0 = \frac{4}{F_0^2}\left(t^2 - x^2 + 1\right). \tag{27}$$

It is useful to define v using

$$v = v(x,t) = -2\frac{\partial^2}{\partial x\,\partial t}\ln[F(x,t)].$$

Then

$$v_0 = \frac{8tx}{F_0^2}.$$

6.1. AN ILLUSTRATIVE PERTURBATION

We now include a perturbation term, $S(x,t) = u_{xt}$ in the left hand side of the above equation, viz. Eq. (26). Here ϵ is small. We can obtain the correct solution, to order ϵ, by writing $F(x,t) = F_0(x,t) - \epsilon\, t\, x$. This leads to:

$$u = \frac{t^2\left(4 - 2\epsilon^2\right) + 4\epsilon t x - 4x^2 + 4}{\left(F_0 - \epsilon\, t x\right)^2}.$$

So this solution, written to order ϵ, is

$$u = u_0 + \epsilon p_b(x,t),$$

where

$$p_b(x,t) = \frac{4\,tx}{F_0^3}\left(3t^2 - x^2 + 3\right), \tag{28}$$

and this solves Eq. (26) to order ϵ. We plot this in Fig. 9. Then

$$\frac{F_0^3}{2\epsilon}(v(x,t) - v_0) = 1 - t^4 + 6t^2x^2 - x^4 = 1 - \left(t^2 - 2tx - x^2\right)\left(t^2 + 2tx - x^2\right). \tag{29}$$

To order ϵ, the constants of the motion, $\int_{-\infty}^{\infty} u(x,t)\,dx = 0$ and $\int_{-\infty}^{\infty} v(x,t)\,dx = 0$ still hold, as does the conservation law $u_t + v_x = 0$.

Now, using regular rogue wave volume definitions (*e.g.* see [19]) after expanding u and v, we find:

$$V_u = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2(x,t)dx\,dt = 1 + \frac{1}{4}\epsilon^2$$

and

$$V_v = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2(x,t)dx\,dt = \frac{1}{3} + \frac{3}{20}\epsilon^2$$

The 'higher order' volume is:

$$V_h = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3(x,t)dx\,dt = 2 + \frac{3}{5}\epsilon^2.$$

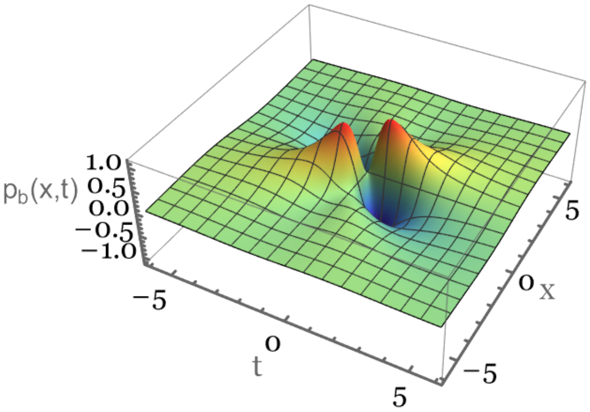

Fig. 9 – *Solution component $p_b(x,t)$ from equation (28) for perturbing function $S(x,t) = u_{xt}$ in equation (26). So the full solution is given by $u = u_0 + \epsilon p_b(x,t)$, where ϵ is small. The maximum is about 1.122, so we need to ensure that* $1.122\epsilon \ll 1$.

For the basic rogue wave of the Boussinesq equation, *viz.* Eq. (27), these quantities have exact values of $1, 1/3$, and 2, respectively [27]. Hence, to the accuracy employed here (i.e. order ϵ), these remain unchanged when the perturbation is added.

In fact, this is not unique, as we can extend it by using:

$$F(x,t) = t^2 + x^2 + 1 - \epsilon \left[x(t + d_2) + j_2 t\right],$$

where d_2, j_2 are arbitrary constants.

This corresponds, to order ϵ, to having

$$p(x,t)\,F_0^3/4 = x(d_2 + t)\left(3t^2 - x^2 + 3\right) + j_2 t\left(t^2 - 3x^2 + 1\right).$$

If we set $d_2 = 0, j_2 = 0$, then the solution reduces to that above, *i.e.* equation (28).

For example, d_2 would be set by the initial condition:

$$u(x,0) = \frac{4\left(1 - x^2\right)}{\left(x^2 + 1\right)^2} + 4\epsilon\, d_2 \frac{x\left(3 - x^2\right)}{\left(x^2 + 1\right)^3}.$$

Further,

$$(v - v_0)F_0^3/(2\epsilon) = -2d_2 t\left(t^2 - 3x^2 + 1\right) + 6t^2 x(j_2 + x) - \left(x^2 + 1\right)\left(2j_2 x + x^2 - 1\right) - t^4.$$

If $d_2 = j_2 = 0$, this reduces to Eq. (29).

7. CONCLUSION

Our aim was to analyze perturbations on the nonlinear Schrödinger and Boussinesq equations. We have done this by determining the actual rogue wave shape changes created by the appearance of various physical perturbing effects and we have plotted these. This work can be important in that it allows assessment of the maximum local amplitude change due to a perturbation that will not significantly disrupt the whole evolution of a rogue wave pattern.

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