LIE GROUP STRUCTURES AND SOLITON SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATION

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This paper investigates the Lie symmetry structure, conservation laws, and novel soliton solutions of a nonlinear Schrodinger equation. We determine the admitted Lie point symmetries and establish the associated Lie algebra generated by $\Gamma_1 = \partial_x$, $\Gamma_2 = \partial_t$, and $\Gamma_3 = x\partial_x + 2t\partial_t$. The algebraic structure is explored through the commutator table, adjoint representation, and classification of the optimal system of subalgebras. Corresponding similarity reductions transform the PDE system into reduced ordinary differential equations, for which approximate solutions are constructed using power series methods. In addition, conservation laws are systematically derived *via* conservation theorem, linking the symmetries to physically meaningful invariants. Novel soliton solutions have been established using the modified generalized Riccati equation mapping method (MGREMM). The results provide a comprehensive symmetry-based framework for the system under study, contributing to the broader understanding of nonlinear coupled PDEs and their analytical properties.

Key words: Lie symmetry; Lie sub algebra; optimal systems; conservation laws; MGREMM; soliton solutions.

1. INTRODUCTION

Nonlinear partial differential equations (PDEs) play a central role in modeling complex phenomena in physics, engineering, and applied sciences. In particular, systems of coupled nonlinear PDEs arise naturally in fluid dynamics, plasma physics, nonlinear optics, and the theory of integrable systems. The mathematical challenges associated with these equations stem from their nonlinear structure and the presence of non-trivial coupling terms, which often prevent the direct application of standard solution techniques [1–15].

In recent decades, Lie group analysis has emerged as one of the most powerful tools for investigating nonlinear PDEs. By exploiting the invariance properties of differential equations under continuous transformation groups, Lie symmetries provide systematic techniques for symmetry reductions, classification of invariant solutions, construction of optimal systems, and analysis of the underlying algebraic structures. Moreover, Lie symmetries play an essential role in establishing connections between integrability, conservation laws, and invariant solutions [16–22].

The use of symmetry methods in nonlinear PDE analysis has a long and rich history. The pioneering works of Sophus Lie established the foundation of continuous symmetries in differential equations [23]. Later developments by Bluman and Kumei [24] expanded these ideas to practical algorithms for symmetry reductions and similarity solutions. Conservation laws, as fundamental structural invariants, were traditionally obtained *via* Noether's theorem; however, the advent of new theorem [25, 26] provided a unifying framework applicable to both variational and non-variational PDE systems. This has enabled a broader class of physical and mathematical models to be studied through conservation law analysis. Recent studies have applied Lie symmetry methods to nonlinear wave equations [27–29], reaction-diffusion systems, and fluid flow models [30–32].

Nonlinear Schrödinger (NLS) equations have emerged as foundational models in the study of nonlinear dispersive wave phenomena. They arise naturally in many physical contexts–such as nonlinear optics, plasma physics, fluid dynamics, and Bose-Einstein condensates–where the interplay between dispersion and nonlinearity dictates the evolution of wave packets [33–39]. Unlike the linear Schrödinger equation, its nonlinear counterpart admits a wealth of dynamical behaviors including self-focusing, modulational instability, and the formation of coherent structures [40–42].

Among the most remarkable solutions of the NLS equation are solitons–localized waveforms that preserve their shape and velocity over long propagation distances. The concept of solitons, initially observed in shallow water dynamics, later found formal grounding in the theory of integrable systems. In the NLS framework, bright solitons arise in focusing media as localized pulses, while dark solitons represent intensity depressions in defocusing media. Their stability and particle-like interactions have made them indispensable in both theory and applications, including optical communications and ultrafast signal processing [43–48].

In this work, we consider the nonlinear Schrodinger equation [49] given by

$$-i\vartheta_t + \vartheta_{xx} + \frac{2|\vartheta_x|^2\vartheta}{1-\vartheta\vartheta^*} = 0. \tag{1}$$

In particular, the classification of Lie algebras and the construction of optimal systems, establishing soliton and exact solutions using efficient integration schemes have proven essential in reducing PDEs to ordinary differential equations (ODEs) for obtaining exact or approximate solutions.

It should be noted that, the specific nonlinear Schrödinger equation studied in this paper has not, to the best of our knowledge, been systematically analyzed from the viewpoint of Lie symmetries, algebra classification, conservation laws as well as presentation of its solution by the employed method. Our contribution therefore extends the scope of symmetry-based analysis to this class of nonlinear Schrödinger equation, providing both theoretical results and methodological tools for future studies.

The paper contributes to the complete Lie symmetry classification and optimal system construction, the derivation of ODE reductions and approximate power-series solutions, the systematic determination of conservation laws *via* Ibragimov's theorem, and presentation of novel soliton solutions. These results contribute to the broader program of symmetry-based methods in nonlinear PDE analysis and provide tools for further investigations of integrability, exact solutions, and qualitative properties of the model.

2. LIE GROUP ANALYSIS

In this Section, we perform the Lie group analysis for the governing equation. We take into account the transformation:

$$\vartheta(x,t) = P(x,t) + iQ(x,t),\ \vartheta^*(x,t) = P(x,t) - iQ(x,t). \tag{2}$$

Substituting Eq. (2) into Eq. (1), we decompose into real and imaginary parts as follows:

$$\begin{aligned} &Q_t - \frac{PP_x^2}{P^2+Q^2-1} - \frac{PQ_x^2}{P^2+Q^2-1} + P_{xx} = 0,\\ &P_t + \frac{QQ_x^2}{P^2+Q^2-1} + \frac{QP_x^2}{P^2+Q^2-1} - Q_{xx} = 0. \end{aligned} \tag{3}$$

The admitted symmetry generators for the equation (3) are

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = \partial_t, \qquad \Gamma_3 = x\partial_x + 2t\partial_t.$$

2.1. COMMUTATORS

The non-vanishing commutation relations are

$$[\Gamma_1, \Gamma_3] = \Gamma_1, \qquad [\Gamma_2, \Gamma_3] = 2\Gamma_2.$$

The commutator table is:

$[\cdot,\cdot]$	Γ_1	Γ_2	Γ_3
Γ_1	0	0	Γ_1
Γ_2	0	0	$2\Gamma_2$
Γ_3	$-\Gamma_1$	$-2\Gamma_2$	0

2.2. ADJOINT REPRESENTATION

The adjoint action is defined as

$$\mathrm{Ad}\big(\exp(\varepsilon X)\big)Y = e^{\varepsilon\,\mathrm{ad}_X}Y, \qquad \mathrm{ad}_X(Y) = [X,Y].$$

From the commutation relations we obtain

$$\mathrm{Ad}\big(\exp(s\Gamma_3)\big)\Gamma_1 = e^{-s}\Gamma_1, \qquad \mathrm{Ad}\big(\exp(s\Gamma_3)\big)\Gamma_2 = e^{-2s}\Gamma_2,$$

$$\mathrm{Ad}\big(\exp(s\Gamma_3)\big)\Gamma_3 = \Gamma_3, \qquad \mathrm{Ad}\big(\exp(a\Gamma_1)\big)\Gamma_3 = \Gamma_3 - a\Gamma_1,$$

$$\mathrm{Ad}\big(\exp(b\Gamma_2)\big)\Gamma_3 = \Gamma_3 - 2b\Gamma_2.$$

The adjoint table is

Ad	Γ_1	Γ_2	Γ_3
$\exp(s\Gamma_3)$	$e^{-s}\Gamma_1$	$e^{-2s}\Gamma_2$	Γ_3
$\exp(a\Gamma_1)$	Γ_1	Γ_2	$\Gamma_3 - a\Gamma_1$
$\exp(b\Gamma_2)$	Γ_1	Γ_2	$\Gamma_3 - 2b\Gamma_2$

2.3. LIE ALGEBRA CLASSIFICATION

The Lie algebra is solvable but non-nilpotent. The ideal

$$I = \mathrm{span}\{\Gamma_1,\Gamma_2\}$$

is Abelian, and the adjoint action of Γ_3 on I is diagonal with eigenvalues 1 and 2. Hence, the algebra is the semidirect product

$$\mathfrak{g} \cong \mathbb{R}^2 \rtimes \mathbb{R},$$

classified in Mubarakzyanov's list as $A^a_{3,5}$ with parameter $a = 2$:

$$[e_1,e_3] = e_1, \qquad [e_2,e_3] = 2e_2.$$

2.4. OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS

Herein, we derive the optimal system of one-dimensional subalgebras.

1. Lie algebra and commutators: The generators are

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = \partial_t, \qquad \Gamma_3 = x\partial_x + 2t\partial_t.$$

A direct calculation gives the nonzero brackets

$$[\Gamma_1,\Gamma_3]=\Gamma_1, \qquad [\Gamma_2,\Gamma_3]=2\Gamma_2.$$

Equivalently,

$$[\Gamma_3,\Gamma_1]=-\Gamma_1, \qquad [\Gamma_3,\Gamma_2]=-2\Gamma_2.$$

2. Adjoint representation: We write the adjoint action as

$$Ad\big(e^{\varepsilon X}\big)Y=e^{\varepsilon\, ad_X}Y, \qquad ad_X(Y)=[X,Y].$$

From the brackets:

$$Ad\big(e^{s\Gamma_3}\big)\Gamma_1=e^{-s}\Gamma_1,\ Ad\big(e^{s\Gamma_3}\big)\Gamma_2=e^{-2s}\Gamma_2,\ Ad\big(e^{s\Gamma_3}\big)\Gamma_3=\Gamma_3,$$

$$Ad\big(e^{a\Gamma_1}\big)\Gamma_3=\Gamma_3-a\Gamma_1,\ Ad\big(e^{b\Gamma_2}\big)\Gamma_3=\Gamma_3-2b\Gamma_2, Ad\big(e^{a\Gamma_1}\big)\Gamma_{1,2}=\Gamma_{1,2},$$

$$Ad\big(e^{b\Gamma_2}\big)\Gamma_{1,2}=\Gamma_{1,2}.$$

Matrix form In the ordered basis $(\Gamma_1,\Gamma_2,\Gamma_3)$,

$$ad_{\Gamma_1}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \quad ad_{\Gamma_2}=\begin{pmatrix}0&0&0\\0&0&2\\0&0&0\end{pmatrix}, \quad ad_{\Gamma_3}=\begin{pmatrix}-1&0&0\\0&-2&0\\0&0&0\end{pmatrix}.$$

Hence

$$Ad\big(e^{s\Gamma_3}\big)=e^{s\,ad_{\Gamma_3}}=\big(e^{-s},e^{-2s},1\big), \quad Ad\big(e^{a\Gamma_1}\big)=I+a\,ad_{\Gamma_1},$$

and

$$Ad\big(e^{b\Gamma_2}\big)=I+b\,ad_{\Gamma_2}.$$

3. Strategy for the optimal system: Two one-dimensional subalgebras $\langle V\rangle$ and $\langle\tilde V\rangle$ are equivalent if $\tilde V=\lambda Ad(g)V$ for some g in the Lie group and $\lambda\neq 0$. Thus we classify orbits of nonzero vectors

$$V=a\Gamma_1+b\Gamma_2+c\Gamma_3 \quad (a,b,c\in\mathbb{R})$$

under the adjoint group, up to nonzero scalar multiples.

We will use the explicit coefficient transformations induced by the basic adjoint actions:

$$Ad\big(e^{a_0\Gamma_1}\big):(a,b,c)\mapsto(a-ca_0,\ b,\ c),$$
$$Ad\big(e^{b_0\Gamma_2}\big):(a,b,c)\mapsto(a,\ b-2cb_0,\ c),$$
$$Ad\big(e^{s\Gamma_3}\big):(a,b,c)\mapsto(e^{-s}a,\ e^{-2s}b,\ c).$$

(Note: multiplying V by $\lambda\neq 0$ does not change the one-dimensional subalgebra.)

4. Orbit simplification by cases:

Case A: $c \neq 0$**.** Start with $V = a\Gamma_1 + b\Gamma_2 + c\Gamma_3$.

(a) Use $Ad\big(e^{a_0\Gamma_1}\big)$ with $a_0 = \dfrac{a}{c}$ to zero the Γ_1 coefficient:

$$(a,b,c) \mapsto (a - ca_0,\ b,\ c) = (0,\ b,\ c).$$

(b) Use $Ad\big(e^{b_0\Gamma_2}\big)$ with $b_0 = \dfrac{b}{2c}$ to zero the Γ_2 coefficient:

$$(0,b,c) \mapsto \big(0,\ b - 2cb_0,\ c\big) = (0,0,c).$$

(c) Since scaling by a nonzero constant does not change the 1D subalgebra, normalize to $c = 1$:

$$V \sim \Gamma_3.$$

Therefore, every V with $c \neq 0$ is equivalent to Γ_3, hence one representative is

$$\langle \Gamma_3 \rangle$$

Case B: $c = 0$**.** Then $V = a\Gamma_1 + b\Gamma_2$.

(a) If $a \neq 0$ and $b = 0$, then $V \sim \Gamma_1$.

(b) If $a = 0$ and $b \neq 0$, then $V \sim \Gamma_2$.

(c) If $a \neq 0$ and $b \neq 0$, act by $Ad\big(e^{s\Gamma_3}\big)$:

$$(a,b,0) \mapsto (e^{-s}a,\ e^{-2s}b,\ 0).$$

The ratio transforms as

$$\frac{b'}{a'} = \frac{e^{-2s}b}{e^{-s}a} = e^{-s}\frac{b}{a}.$$

Choose $s = \ln\left|\dfrac{b}{a}\right|$ to obtain $\dfrac{b'}{a'} = \pm 1$; then multiply V by -1 if needed (overall scalar) to get $b'/a' = 1$. Rescaling gives the canonical representative

$$V \sim \Gamma_1 + \Gamma_2.$$

Thus, for $c = 0$ we obtain three inequivalent classes:

$$\langle \Gamma_1 \rangle, \qquad \langle \Gamma_2 \rangle, \qquad \langle \Gamma_1 + \Gamma_2 \rangle.$$

5. Optimal system in summary: Collecting both cases, an optimal system of one-dimensional subalgebras is

$$\langle \Gamma_1 \rangle, \quad \langle \Gamma_2 \rangle, \quad \langle \Gamma_3 \rangle, \quad \langle \Gamma_1 + \Gamma_2 \rangle.$$

6. Consistency checks

- The adjoint orbits of Γ_1 and Γ_2 cannot intersect: the ratio $\dfrac{b}{a}$ for $a\Gamma_1 + b\Gamma_2$ changes by a positive factor e^{-s} under $Ad(e^{s\Gamma_3})$ and cannot turn a pure Γ_1 vector into a pure Γ_2 one.
- Any vector with $c \neq 0$ can be shifted (by $Ad(e^{a_0\Gamma_1})$ and $Ad(e^{b_0\Gamma_2})$) to remove its Γ_1 and Γ_2 components, leaving only Γ_3.
- The list contains exactly one representative from each distinct orbit type and is minimal.

3. REDUCTIONS TO ODES

In this Section, using the results obtained in the optimal system, we reduce the PDEs to ODEs and establish power series solution of the reduced equations.

(A) Translation in x: $\langle\Gamma_1\rangle$ Ansatz: $P = P(t)$, $Q = Q(t)$. The system reduces to

$$P_t = 0, \qquad Q_t = 0,$$

so the invariant solutions are constants.

(B) Translation in t: $\langle\Gamma_2\rangle$ Ansatz: $P = P(x)$, $Q = Q(x)$. The reduced ODE system is

$$\begin{aligned} P_{xx} - \frac{P(P_x^2 + Q_x^2)}{P^2 + Q^2 - 1} &= 0,\\ -Q_{xx} + \frac{Q(P_x^2 + Q_x^2)}{P^2 + Q^2 - 1} &= 0. \end{aligned}$$

(C) Scaling symmetry: $\langle\Gamma_3\rangle$ Invariant variable: $\zeta = \frac{t}{x^2}$,
ansatz $P(x,t) = F(\zeta)$, $Q(x,t) = G(\zeta)$. The reduced ODE system is

$$\begin{aligned} G'(\zeta) - \frac{4\zeta^2 F\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} + 4\zeta^2 F''(\zeta) + 6\zeta F'(\zeta) &= 0,\\ F'(\zeta) + \frac{4\zeta^2 G\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} - 4\zeta^2 G''(\zeta) - 6\zeta G'(\zeta) &= 0. \end{aligned}$$

(D) Travelling-wave symmetry: $\langle\Gamma_1+\Gamma_2\rangle$ Invariant variable: $\xi=x-t$, ansatz $P(x,t)=F(\xi)$, $Q(x,t)=G(\xi)$. The reduced ODE system is

$$-G'(\xi)-\frac{F\left(F'^2+G'^2\right)}{F^2+G^2-1}+F''(\xi)=0,$$

$$-F'(\xi)+\frac{G\left(F'^2+G'^2\right)}{F^2+G^2-1}-G''(\xi)=0.$$

3.1. POWER SERIES SOLUTIONS

In each reduction we expand the dependent variables in a Taylor series about the expansion point (taken to be the origin of the reduced independent variable). Below we denote the expansion variable and series coefficients for each reduction as follows:

- Reduction (2) (time-translation Γ_2): independent variable x,

  $$\begin{aligned} P(x) &= p_0+p_1x+p_2x^2+p_3x^3+\mathcal{O}(x^4),\\ Q(x) &= q_0+q_1x+q_2x^2+q_3x^3+\mathcal{O}(x^4). \end{aligned}$$

  Let $D_0:=p_0^2+q_0^2-1$ (assumed nonzero so the denominators below are defined).

- Reduction (3) (scaling Γ_3): independent variable $\zeta=\dfrac{t}{x^2}$,

  $$\begin{aligned} P(x,t) &= F(\zeta)=f_0+f_1\zeta+f_2\zeta^2+f_3\zeta^3+\mathcal{O}(\zeta^4),\\ Q(x,t) &= G(\zeta)=g_0+g_1\zeta+g_2\zeta^2+g_3\zeta^3+\mathcal{O}(\zeta^4), \end{aligned}$$

  with $D_0:=f_0^2+g_0^2-1$.

- Reduction (4) (travelling $\Gamma_1+\Gamma_2$): independent variable $\xi=x-t$,

  $$\begin{aligned} P(x,t) &= F(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3+\mathcal{O}(\xi^4),\\ Q(x,t) &= G(\xi)=b_0+b_1\xi+b_2\xi^2+b_3\xi^3+\mathcal{O}(\xi^4), \end{aligned}$$

  with $D_0:=a_0^2+b_0^2-1$.

We now give the substitution, coefficient matching, and explicit formulas for the first nontrivial coefficients in each case.

REDUCTION (2): $P = P(x),\ Q = Q(x)$ (TIME-TRANSLATION)

The reduced ODE system is

$$P_{xx} - \frac{P\big(P_x^2 + Q_x^2\big)}{P^2 + Q^2 - 1} = 0,$$

$$-Q_{xx} + \frac{Q\big(P_x^2 + Q_x^2\big)}{P^2 + Q^2 - 1} = 0.$$

We substitute the series

$$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots, \qquad Q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots,$$

compute P_x, P_{xx}, Q_x, Q_{xx}, expand each equation in powers of x and equate coefficients to zero. The algebra yields the following recursion (we display the first nontrivial coefficients):

$$\text{At } x^0: \quad -p_0(p_1^2 + q_1^2) + 2p_2 D_0 = 0 \quad \Longrightarrow \quad p_2 = \frac{p_0\,(p_1^2 + q_1^2)}{2\,D_0},$$

$$\text{At } x^0 \text{ (second equation):} \quad q_0(p_1^2 + q_1^2) - 2\,q_2\,D_0 = 0 \quad \Longrightarrow \quad q_2 = \frac{q_0\,(p_1^2 + q_1^2)}{2\,D_0}.$$

Proceeding one order higher (equating the coefficient of x^1 to zero) gives

$$6\,D_0\,p_3 - p_1\,(p_1^2 + q_1^2) = 0 \quad \Longrightarrow \quad p_3 = \frac{p_1\,(p_1^2 + q_1^2)}{6\,D_0},$$

$$6\,D_0\,q_3 - q_1\,(p_1^2 + q_1^2) = 0 \quad \Longrightarrow \quad q_3 = \frac{q_1\,(p_1^2 + q_1^2)}{6\,D_0}.$$

Therefore, up to cubic order,

$$P(x) = p_0 + p_1 x + \frac{p_0(p_1^2 + q_1^2)}{2D_0}\,x^2 + \frac{p_1(p_1^2 + q_1^2)}{6D_0}\,x^3 + \mathcal{O}(x^4),$$

$$Q(x) = q_0 + q_1 x + \frac{q_0(p_1^2 + q_1^2)}{2D_0}\,x^2 + \frac{q_1(p_1^2 + q_1^2)}{6D_0}\,x^3 + \mathcal{O}(x^4),$$

with $D_0 = p_0^2 + q_0^2 - 1$. The free parameters are p_0, q_0, p_1, q_1 (they are the Cauchy data at $x = 0$).

REDUCTION (3): SCALING Γ_3 (VARIABLE $\zeta = t/x^2$)

The reduced system (with prime denoting $d/d\zeta$) is

$$G'(\zeta) - \frac{4\zeta^2 F\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} + 4\zeta^2 F''(\zeta) + 6\zeta F'(\zeta) = 0,$$

$$F'(\zeta) + \frac{4\zeta^2 G\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} - 4\zeta^2 G''(\zeta) - 6\zeta G'(\zeta) = 0.$$

We assume

$$F(\zeta) = f_0 + f_1\zeta + f_2\zeta^2 + f_3\zeta^3 + \cdots, \qquad G(\zeta) = g_0 + g_1\zeta + g_2\zeta^2 + g_3\zeta^3 + \cdots,$$

and set $D_0 := f_0^2 + g_0^2 - 1$ (again assume $D_0 \neq 0$). We expand the equations in powers of ζ and match the coefficients.

The coefficient matching gives immediate simple relations at the lowest orders:

$$\begin{aligned}
&\zeta^0 \text{ terms:} \quad g_1 = 0, \qquad f_1 = 0,\\
&\zeta^1 \text{ terms:} \quad 6f_1 + 2g_2 = 0 \;\Rightarrow\; g_2 = 0, \qquad 2f_2 - 6g_1 = 0 \;\Rightarrow\; f_2 = 0,\\
&\zeta^2 \text{ terms:} \quad 3g_3\, D_0 = 0 \;\Rightarrow\; g_3 = 0, \qquad 3f_3\, D_0 = 0 \;\Rightarrow\; f_3 = 0.
\end{aligned}$$

Hence, up to cubic order the series truncates to constants at this regular expansion point:

$$F(\zeta) = f_0 + \mathcal{O}(\zeta^4), \qquad G(\zeta) = g_0 + \mathcal{O}(\zeta^4),$$

and the first nonzero derivatives at the origin are forced to vanish by the differential relations. (This indicates that a nontrivial small-ζ behaviour may either require higher-order terms, non-analytic expansions, or different expansion points; but the regular Taylor expansion about $\zeta = 0$ yields constant leading terms with vanishing low-order derivatives.)

REDUCTION (4): TRAVELLING-WAVE $\Gamma_1 + \Gamma_2$ (VARIABLE $\xi = x - t$)

The reduced travelling ODE system (prime denotes $d/d\xi$) is

$$-G'(\xi) - \frac{F\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} + F''(\xi) = 0,$$

$$-F'(\xi) + \frac{G\left(F'^2 + G'^2\right)}{F^2 + G^2 - 1} - G''(\xi) = 0.$$

We set

$$F(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots, \qquad G(\xi) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \cdots,$$

and $D_0 := a_0^2 + b_0^2 - 1$ (we assume $D_0 \neq 0$). We substitute and match the coefficients. The lowest-order relations give

$$\text{At } \xi^0: \quad -a_0(a_1^2 + b_1^2) + (2a_2 - b_1)D_0 = 0 \quad \Longrightarrow \quad a_2 = \frac{b_1}{2} + \frac{a_0(a_1^2 + b_1^2)}{2D_0},$$

$$\text{At } \xi^0: \text{ (second equation)} \quad b_0(a_1^2 + b_1^2) - (a_1 + 2b_2)D_0 = 0$$
$$\Longrightarrow \quad b_2 = \frac{b_0(a_1^2 + b_1^2)}{2D_0} - \frac{a_1}{2}.$$

Equating the coefficients at order ξ^1 yields linear equations for a_3, b_3. Solving these gives

$$a_3 = \frac{-a_0^2 a_1 + a_1^3 + a_1^2 b_0 - a_1 b_0^2 + a_1 b_1^2 + a_1 + b_0 b_1^2}{6\,D_0},$$

$$b_3 = \frac{-a_0^2 b_1 - a_0 a_1^2 - a_0 b_1^2 + a_1^2 b_1 - b_0^2 b_1 + b_1^3 + b_1}{6\,D_0}.$$

Thus, up to cubic order,

$$F(\xi) = a_0 + a_1\xi + \Big(\frac{b_1}{2} + \frac{a_0(a_1^2 + b_1^2)}{2D_0}\Big)\xi^2 + \; a_3\,\xi^3 + \mathcal{O}(\xi^4),$$

$$G(\xi) = b_0 + b_1\xi + \Big(\frac{b_0(a_1^2 + b_1^2)}{2D_0} - \frac{a_1}{2}\Big)\xi^2 + \; b_3\,\xi^3 + \mathcal{O}(\xi^4),$$

with a_3, b_3 as displayed above and $D_0 = a_0^2 + b_0^2 - 1$.

REMARKS

1. In each case the lowest coefficients p_0, q_0, p_1, q_1 (or a_0, b_0, a_1, b_1, etc.) are free constants representing initial data at the expansion point. The higher coefficients are determined algebraically in terms of them, as shown.

2. The scaling reduction (3) yielded vanishing low-order derivatives in a regular Taylor expansion about $\zeta = 0$; that indicates either constant leading behaviour near $\zeta = 0$ or that nontrivial solutions may require expansion about a different point or non-analytic series such as fractional powers, a common occurrence when the independent variable appears multiplied by derivatives as in this scaling reduction.

4. CONSERVATION LAWS

We examine the conservation laws of the system under consideration. Consider

$$F_1 := Q_t - \frac{P(P_x^2+Q_x^2)}{P^2+Q^2-1} + P_{xx} = 0, \tag{4}$$

$$F_2 := P_t + \frac{Q(P_x^2+Q_x^2)}{P^2+Q^2-1} - Q_{xx} = 0, \tag{5}$$

with Lie point symmetries

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = \partial_t, \qquad \Gamma_3 = x\partial_x + 2t\,\partial_t.$$

We introduce the adjoint fields $\Phi(x,t)$ and $\Psi(x,t)$ and the *formal Lagrangian*

$$\mathcal{L} = \Phi\, F_1 + \Psi\, F_2.$$

For compactness we denote

$$D := P^2+Q^2-1, \qquad S := P_x^2+Q_x^2, \qquad M := -\Phi P + \Psi Q.$$

The adjoint system $F_1^* = 0,\ F_2^* = 0$ is obtained from the Euler operators

$$F_P^* := \frac{\delta\mathcal{L}}{\delta P} = 0, \qquad F_Q^* := \frac{\delta\mathcal{L}}{\delta Q} = 0,$$

where

$$\frac{\delta\mathcal{L}}{\delta u} = \frac{\partial\mathcal{L}}{\partial u} - D_t\left(\frac{\partial\mathcal{L}}{\partial u_t}\right) - D_x\left(\frac{\partial\mathcal{L}}{\partial u_x}\right) + D_x^2\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right), \quad u \in \{P,Q\}.$$

The needed partial derivatives are

$$\frac{\partial\mathcal{L}}{\partial P_t} = \Psi, \qquad \frac{\partial\mathcal{L}}{\partial Q_t} = \Phi, \qquad \frac{\partial\mathcal{L}}{\partial P_{xx}} = \Phi, \qquad \frac{\partial\mathcal{L}}{\partial Q_{xx}} = -\Psi,$$

$$\frac{\partial\mathcal{L}}{\partial P_x} = \frac{2P_x M}{D}, \qquad \frac{\partial\mathcal{L}}{\partial Q_x} = \frac{2Q_x M}{D},$$

$$\frac{\partial\mathcal{L}}{\partial P} = -\Phi\left(\frac{S}{D} - \frac{2P^2S}{D^2}\right) - \frac{2\Psi QPS}{D^2}, \qquad \frac{\partial\mathcal{L}}{\partial Q} = \frac{2\Phi PQS}{D^2} + \Psi\left(\frac{S}{D} - \frac{2Q^2S}{D^2}\right).$$

Hence

$$F_P^* = \frac{\partial\mathcal{L}}{\partial P} - D_t(\Psi) - D_x\left(\frac{2P_x M}{D}\right) + D_x^2(\Phi) = 0,$$

$$F_Q^* = \frac{\partial\mathcal{L}}{\partial Q} - D_t(\Phi) - D_x\left(\frac{2Q_x M}{D}\right) - D_x^2(\Psi) = 0.$$

For a Lie symmetry

$$X = \xi^t\partial_t + \xi^x\partial_x + \eta^P\partial_P + \eta^Q\partial_Q,$$

we define the characteristics

$$W^P := \eta^P - \xi^t P_t - \xi^x P_x, \qquad W^Q := \eta^Q - \xi^t Q_t - \xi^x Q_x.$$

Since our generators have no ∂_P, ∂_Q components, we have $\eta^P = \eta^Q = 0$.

For equations up to u_{xx} only, Ibragimov's conserved vector $C = (C^t, C^x)$ is

$$\begin{aligned} C^t &= \xi^t \mathcal{L} + W^P \frac{\partial \mathcal{L}}{\partial P_t} + W^Q \frac{\partial \mathcal{L}}{\partial Q_t}, \\ C^x &= \xi^x \mathcal{L} + W^P \left(\frac{\partial \mathcal{L}}{\partial P_x} - D_x \frac{\partial \mathcal{L}}{\partial P_{xx}} \right) + (D_x W^P) \frac{\partial \mathcal{L}}{\partial P_{xx}} \\ &\quad + W^Q \left(\frac{\partial \mathcal{L}}{\partial Q_x} - D_x \frac{\partial \mathcal{L}}{\partial Q_{xx}} \right) + (D_x W^Q) \frac{\partial \mathcal{L}}{\partial Q_{xx}}. \end{aligned}$$

With the derivatives above this becomes the convenient general form

$$\begin{aligned} C^t &= \xi^t \mathcal{L} + W^P \Psi + W^Q \Phi, \\ C^x &= \xi^x \mathcal{L} + W^P \left(\frac{2P_x M}{D} - \Phi_x \right) + (D_x W^P) \Phi \\ &\quad + W^Q \left(\frac{2Q_x M}{D} + \Psi_x \right) - (D_x W^Q) \Psi. \end{aligned}$$

By construction,

$$D_t C^t + D_x C^x \equiv W^P F_P^* + W^Q F_Q^* + (\text{terms proportional to } F_1, F_2),$$

so $D_t C^t + D_x C^x = 0$ on joint solutions of Eqs. (4), (5) and $F_P^* = F_Q^* = 0$.

CONSERVATION LAWS FOR EACH SYMMETRY

(i) Spatial translation $\Gamma_1 = \partial_x$. Here $\xi^x = 1$, $\xi^t = 0$, and

$$W^P = -P_x, \qquad W^Q = -Q_x, \qquad D_x W^P = -P_{xx}, \qquad D_x W^Q = -Q_{xx}.$$

Thus

$$\begin{aligned} C^t_{\Gamma_1} &= -P_x \Psi - Q_x \Phi, \\ C^x_{\Gamma_1} &= \mathcal{L} - P_x \left(\frac{2P_x M}{D} - \Phi_x \right) - P_{xx} \Phi - Q_x \left(\frac{2Q_x M}{D} + \Psi_x \right) + Q_{xx} \Psi. \end{aligned}$$

On the original equations $F_1 = F_2 = 0$, the $\mathcal{L}$ term vanishes.

(ii) Time translation $\Gamma_2 = \partial_t$. Here $\xi^t = 1$, $\xi^x = 0$, and

$$W^P = -P_t, \qquad W^Q = -Q_t, \qquad D_x W^P = -P_{tx}, \qquad D_x W^Q = -Q_{tx}.$$

Hence

$$C^t_{\Gamma_2} = \mathcal{L} - P_t \Psi - Q_t \Phi,$$

$$C^x_{\Gamma_2} = -P_t\left(\frac{2P_x M}{D} - \Phi_x\right) - P_{tx}\Phi - Q_t\left(\frac{2Q_x M}{D} + \Psi_x\right) + Q_{tx}\Psi.$$

Again $\mathcal{L}$ drops out on $F_1 = F_2 = 0$.

(iii) Scaling $\Gamma_3 = x\partial_x + 2t\partial_t$. Here $\xi^x = x,\ \xi^t = 2t$, and

$$W^P = -2t P_t - x P_x, \qquad D_x W^P = -2t P_{tx} - x P_{xx} - P_x,$$

$$W^Q = -2t Q_t - x Q_x, \qquad D_x W^Q = -2t Q_{tx} - x Q_{xx} - Q_x.$$

Therefore

$$C^t_{\Gamma_3} = 2t\mathcal{L} + (-2t P_t - x P_x)\Psi + (-2t Q_t - x Q_x)\Phi,$$

$$C^x_{\Gamma_3} = x\mathcal{L} + (-2t P_t - x P_x)\left(\frac{2P_x M}{D} - \Phi_x\right) + (-2t P_{tx} - x P_{xx} - P_x)\Phi$$
$$+ (-2t Q_t - x Q_x)\left(\frac{2Q_x M}{D} + \Psi_x\right) - (-2t Q_{tx} - x Q_{xx} - Q_x)\Psi.$$

On $F_1 = F_2 = 0$ the terms with $\mathcal{L}$ vanish.

5. THE CHARACTERIZATION OF THE SPECIFIED ANALYTICAL METHODOLOGY

This Section presents the essential ideas of the MGREMM [50]. The fundamental stages of the MGREMM are delineated below:

Step 1: We consider the following expression as indicative of the comprehensive framework of nonlinear partial differential equations (NPDEs)

$$\Omega(\vartheta, \vartheta_x, \vartheta_t, \vartheta_{xt}, \vartheta_{xx}, ...) = 0, \tag{6}$$

where Ω denotes a set of polynomials that encompasses $\vartheta(x,t)$ along with its derivatives.

We take into account

$$\vartheta(x,t) = \phi(\varsigma)e^{i(-\beta x + \gamma t + \Theta_0)}, \varsigma = \alpha x - vt, \tag{7}$$

whereas $e^{i(-\beta x + \gamma t + \Theta_0)}$ denotes the phase component, and v signifies the wave velocity. In the phase components, β represents soliton frequency, γ denotes wave number, and Θ_0 signifies the phase center. By employing Eq. (7) in Eq. (6), a nonlinear ordinary differential equation (NODE) is derived as:

$$G(\phi, \phi', \phi'', ...) = 0, \tag{8}$$

In this context, G represents a polynomial that includes $\phi(\varsigma)$, $\phi'(\varsigma)$, $\phi''(\varsigma)$, and so forth, all of which are total derivatives. The prime symbol $'$ indicates the operation $\frac{d}{d\varsigma}$.

Step 2: Here is how to write the solution to Eq. (8):

$$\phi(\varsigma) = a_0 + \sum_{i=1}^{m} a_i (K(\varsigma))^i + \sum_{i=1}^{m} b_i \left(\frac{K'(\varsigma)}{K(\varsigma)} \right)^i, \tag{9}$$

where

$$K'(\varsigma) = \delta_0 + \delta_1 K(\varsigma) + \delta_2 K^2(\varsigma), \tag{10}$$

δ_1, δ_2, and δ_3 are real constants and $\sigma = \delta_1^2 - 4\delta_0\delta_2$. The value of m can be determined using the balance principle, which establishes a relationship between the highest derivative and the nonlinear term.

Step 3: By utilizing Eq. (9) within Eq. (10) and performing some calculations, all terms can be collected to obtain an algebraic equation. Setting each coefficient of the equation generates a system of equations.

Step 4: By solving the obtained system, the unknown parameters can be determined and substituted into Eq. (9). This leads to obtaining the anticipated solutions for Eq. (1). The roots regarding the parameters δ_1, δ_2, and δ_3 of Eq. (8) are:

Family 1: When $\sigma = \delta_1^2 - 4\delta_0\delta_2 > 0$, and $\delta_1\delta_2 \neq 0$ (or $\delta_0\delta_2 \neq 0$):

$$K_1(\varsigma) = -\frac{1}{2\delta_2}\left(\delta_1 + \sqrt{\sigma}\tanh\left(\frac{\sqrt{\sigma}}{2}\varsigma \right) \right), \tag{11}$$

$$K_2(\varsigma) = -\frac{1}{2\delta_2}\left(\delta_1 + \sqrt{\sigma}\coth\left(\frac{\sqrt{\sigma}}{2}\varsigma \right) \right), \tag{12}$$

$$K_3(\varsigma) = -\frac{1}{2\delta_2}\left(\delta_1 + \sqrt{\sigma}\left(\tanh\left(\sqrt{\sigma}\varsigma \right) \pm i\,\mathrm{sech}\left(\sqrt{\sigma}\varsigma \right) \right) \right), \tag{13}$$

$$K_4(\varsigma) = -\frac{1}{2\delta_2}\left(\delta_1 + \sqrt{\sigma}\left(\coth\left(\sqrt{\sigma}\varsigma \right) \pm \mathrm{csch}\left(\sqrt{\sigma}\varsigma \right) \right) \right), \tag{14}$$

$$K_5(\varsigma) = -\frac{1}{4\delta_2}\left(2\delta_1 + \sqrt{\sigma}\left(\tanh\left(\frac{\sqrt{\sigma}}{4}\varsigma \right) + \coth\left(\frac{\sqrt{\sigma}}{4}\varsigma \right) \right) \right), \tag{15}$$

$$\begin{aligned} K_6(\varsigma) &= \frac{\sqrt{(A^2+B^2)\left(\delta_1^2 - 4\delta_0\delta_2\right)} - A\sqrt{\delta_1^2 - 4\delta_0\delta_2}\cosh\left(\sqrt{\delta_1^2 - 4\delta_0\delta_2}\varsigma \right)}{2\delta_2\left(B + A\sinh\left(\sqrt{\delta_1^2 - 4\delta_0\delta_2}\varsigma \right) \right)} \\ &- \frac{\delta_1}{2\delta_2}, \end{aligned} \tag{16}$$

$$K_7(\varsigma) = -\frac{\sqrt{\left(-A^2+B^2\right)\left(\delta_1^2-4\delta_0\delta_2\right)}+A\sqrt{\delta_1^2-4\delta_0\delta_2}\sinh\left(\sqrt{\delta_1^2-4\delta_0\delta_2}\varsigma\right)}{B+A\cosh\left(\sqrt{\delta_1^2-4\delta_0\delta_2}\varsigma\right)} - \frac{\delta_1}{2\delta_2}, \tag{17}$$

$$K_8(\varsigma) = \frac{2\delta_0\cosh\left(\sqrt{\sigma}\varsigma\right)}{\sqrt{\sigma}\sinh\left(\sqrt{\sigma}\varsigma\right)-\delta_1\cosh\left(\sqrt{\sigma}\varsigma\right)}, \tag{18}$$

$$K_9(\varsigma) = \frac{2\delta_0\sinh\left(\sqrt{\sigma}\varsigma\right)}{\sqrt{\sigma}\cosh\left(\sqrt{\sigma}\varsigma\right)-\delta_1\sinh\left(\sqrt{\sigma}\varsigma\right)}, \tag{19}$$

$$K_{10}(\varsigma) = \frac{2\delta_0\cosh\left(\sqrt{\sigma}\varsigma\right)}{\sqrt{\sigma}\sinh\left(\sqrt{\sigma}\varsigma\right)-\delta_1\cosh\left(\sqrt{\sigma}\varsigma\right)\pm i\sqrt{\sigma}}, \tag{20}$$

$$K_{11}(\varsigma) = \frac{2\delta_0\sinh\left(\sqrt{\sigma}\varsigma\right)}{\sqrt{\sigma}\cosh\left(\sqrt{\sigma}\varsigma\right)-\delta_1\sinh\left(\sqrt{\sigma}\varsigma\right)\pm\sqrt{\sigma}}, \tag{21}$$

$$K_{12}(\varsigma) = \frac{4\delta_0\sinh\left(\frac{\sqrt{\sigma}}{4}\varsigma\right)\cosh\left(\frac{\sqrt{\sigma}}{4}\varsigma\right)}{2\sqrt{\sigma}\cosh^2\left(\frac{\sqrt{\sigma}}{4}\varsigma\right)-2\delta_1\sinh\left(\frac{\sqrt{\sigma}}{4}\varsigma\right)\cosh\left(\frac{\sqrt{\sigma}}{4}\varsigma\right)-\sqrt{\sigma}}. \tag{22}$$

Family 2: When $\delta_1^2-4\delta_0\delta_2<0$, and $\delta_1\delta_2\neq 0$ (or $\delta_0\delta_2\neq 0$):

$$K_{13}(\varsigma) = \frac{1}{2\,\delta_2}+\left(-\delta_1+\sqrt{-\sigma}\tan\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)\right), \tag{23}$$

$$K_{14}(\varsigma) = -\frac{1}{2\,\delta_2}+\left(\delta_1+\sqrt{-\sigma}\cot\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)\right), \tag{24}$$

$$K_{15}(\varsigma) = \frac{1}{2\,\delta_2}+\left(-\delta_1+\sqrt{-\sigma}\tan\left(\sqrt{-\sigma}\varsigma\right)\pm\sec\left(\sqrt{-\sigma}\varsigma\right)\right), \tag{25}$$

$$K_{16}(\varsigma) = \frac{1}{2\,\delta_2}+\left(-\delta_1+\sqrt{-\sigma}\cot\left(\sqrt{-\sigma}\varsigma\right)\pm\csc\left(\sqrt{-\sigma}\varsigma\right)\right), \tag{26}$$

$$K_{17}(\varsigma) = \frac{1}{4\,\delta_2}+\left(-2\delta_1+\sqrt{-\sigma}\left(\tan\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)-\cot\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)\right)\right), \tag{27}$$

$$K_{18}(\varsigma) = \frac{1}{2\delta_2}\left(\frac{-\delta_1+\sqrt{\sigma\left(B^2-A^2\right)}-A\sqrt{-\sigma}\cos\left(\sqrt{-\sigma}\varsigma\right)}{A\sin\left(\sqrt{-\sigma}\varsigma\right)+B}\right), \tag{28}$$

$$K_{19}(\varsigma) = \frac{2\,\delta_0\cos\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)}{\sqrt{-\sigma}\sin\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)+\delta_1\cos\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)}, \tag{29}$$

$$K_{20}(\varsigma)=\frac{2\,\delta_0\sin\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)}{-\delta_1\sin\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)+\sqrt{-\sigma}\cos\left(\frac{\sqrt{-\sigma}}{2}\varsigma\right)},\tag{30}$$

$$K_{21}(\varsigma)=\frac{2\,\delta_0\cos\left(\sqrt{-\sigma}\varsigma\right)}{\sqrt{-\sigma}\sin\left(\sqrt{-\sigma}\varsigma\right)+\delta_1\cos\left(\sqrt{-\sigma}\varsigma\right)\pm\sqrt{-\sigma}},\tag{31}$$

$$K_{22}(\varsigma)=\frac{2\,\delta_0\sin\left(\sqrt{-\sigma}\varsigma\right)}{\sqrt{-\sigma}\cos\left(\sqrt{-\sigma}\varsigma\right)+\delta_1\sin\left(\sqrt{-\sigma}\varsigma\right)\pm\sqrt{-\sigma}},\tag{32}$$

$$K_{23}(\varsigma)=\frac{4\,\delta_0\sin\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)}{2\sqrt{-\sigma}\cos^2\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)-2\delta_1\sin\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)\cos\left(\frac{\sqrt{-\sigma}}{4}\varsigma\right)-\sqrt{-\sigma}}.\tag{33}$$

Family 3: When $\delta_0=0$ and $\delta_1\delta_2\neq 0$:

$$K_{24}(\varsigma)=-\frac{\delta_1\varsigma_0}{\delta_2\left(\varsigma_0+\cosh\left(\delta_1\varsigma\right)-\sinh\left(\delta_1\varsigma\right)\right)},\tag{34}$$

$$K_{25}(\varsigma)=-\frac{\delta_1\left(\cosh\left(\delta_1\varsigma\right)-\sinh\left(\delta_1\varsigma\right)\right)}{\delta_2\left(\varsigma_0+\cosh\left(\delta_1\varsigma\right)-\sinh\left(\delta_1\varsigma\right)\right)}.\tag{35}$$

5.1. IMPLEMENTATION OF THE MODIFIED GENERALIZED RICCATI EQUATION MAPPING METHOD

This Section endeavors to formulate optical solutions to equation Eq. (1) through the application of the stated method. To get the solutions of Eq. (1), substitute Eq. (7) into Eq. (1) and then separate the resulting expressions into real and imaginary components, yielding a pair of equations as follows:

$$\left(\gamma-\beta^2\right)\phi+\left(\beta^2-\gamma\right)\phi^3+2\alpha^2\phi\phi'^2+\alpha^2\phi''-\alpha^2\phi^2\phi''=0,\tag{36}$$

and

$$v=2\alpha\beta.\tag{37}$$

By equating ϕ'' with ϕ^3 in Eq. (36), we determine the balance number $m=1$. Substituting this value into Eq. (9) leads to the derivation of the subsequent solution formula:

$$\phi(\varsigma)=a_0+a_1K(\varsigma)+b_1\frac{K'(\varsigma)}{K(\varsigma)}.\tag{38}$$

Subsequently, replace the solution of Eq. (38) in Eq. (36) and identify the coefficients of the various powers; this results in the following system of equations:

$$\begin{array}{rcl}
K^0(\varsigma) & = & -2a_0b_1^2\alpha^2\delta_0^4 - b_1^3\alpha^2\delta_0^4\delta_1, \\
K^1(\varsigma) & = & 2b_1\alpha^2\delta_0^3 - 2a_0^2b_1\alpha^2\delta_0^3 + b_1^3\beta^2\delta_0^3 - b_1^3\gamma\delta_0^3 - 6a_1b_1^2\alpha^2\delta_0^4 - 6a_0b_1^2\alpha^2\delta_0^3\delta_1 \\
 & - & 3b_1^3\alpha^2\delta_0^3\delta_1^2 - 4b_1^3\alpha^2\delta_0^4\delta_2, \\
K^2(\varsigma) & = & 3a_0b_1^2\beta^2\delta_0^2 - 3a_0B_1^2\gamma\delta_0^2 - 8a_0a_1b_1\alpha^2\delta_0^3 + 3b_1\alpha^2\delta_0^2\delta_1 - 3a_0^2b_1\alpha^2\delta_0^2\delta_1 \\
 & + & 3b_1^3\beta^2\delta_0^2\delta_1 - 3b_1^3\gamma\delta_0^2\delta_1 - 19a_1b_1^2\alpha^2\delta_0^3\delta_1 - 6a_0b_1^2\alpha^2\delta_0^2\delta_1^2 - 3b_1^3\alpha^2\delta_0^2\delta_1^3 \\
 & - & 8a_0b_1^2\alpha^2\delta_0^3\delta_2 - 16b_1^3\alpha^2\delta_0^3\delta_1\delta_2, \\
 & \vdots & \\
K^8(\varsigma) & = & a_1^3\alpha^2\delta_1\delta_2 - 2a_0a_1^2\alpha^2\delta_2^2 + a_1^2b_1\alpha^2\delta_1\delta_2^2 - 4a_0a_1b_1\alpha^2\delta_2^3 - a_1b_1^2\alpha^2\delta_1\delta_2^3 \\
 & - & 2a_0b_1^2\alpha^2\delta_2^4 - b_1^3\alpha^2\delta_1\delta_2^4.
\end{array} \tag{39}$$

The results that are obtained upon solving the algebraic system presented in Eq. (39) are as follows:

$$a_0 = \frac{\delta_1}{\sqrt{\delta_1^2 - 4\delta_0\delta_2}}, a_1 = \frac{2\delta_2}{\sqrt{\delta_1^2 - 4\delta_0\delta_2}}, b_1 = -\frac{2}{\sqrt{\delta_1^2 - 4\delta_0\delta_2}}, \gamma = \beta^2. \tag{40}$$

Now multiple distinct solutions by considering Eq. (11) till Eq. (35) are generated. By taking into account Eq. (40) with Eq. (36) and Eq. (37), a variety of exact solutions for the model under study are obtained as follows:

The following solutions are constructed when $\sigma = \delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1\delta_2 \neq 0$ (or $\delta_0\delta_2 \neq 0$) and the solutions in the family 1 are considered:

By employing Eq. (11) in conjunction with Eq. (38) within the framework of Eq. (1), one can derive the rational solutions of the hyperbolic function, as illustrated in Fig. 1:

$$\begin{array}{rcl}
\vartheta_1(x,t) & = & \mathrm{e}^{\mathrm{i}\left(-\beta x + \beta^2 t + \theta_0\right)} \\
 & \times & \dfrac{\left(-\delta_1^2 + 4\delta_0\delta_2 - \delta_1\sqrt{\delta_1^2 - 4\delta_0\delta_2}\tanh\left(\frac{1}{2}\alpha(x - 2\beta t)\sqrt{\delta_1^2 - 4\delta_0\delta_2}\right)\right)}{\sqrt{\delta_1^2 - 4\delta_0\delta_2}\left(\delta_1 + \sqrt{\delta_1^2 - 4\delta_0\delta_2}\tanh\left(\frac{1}{2}\alpha(x - 2\beta t)\sqrt{\delta_1^2 - 4\delta_0\delta_2}\right)\right)}.
\end{array} \tag{41}$$

Plugging Eq. (12) along with Eq. (38), yields

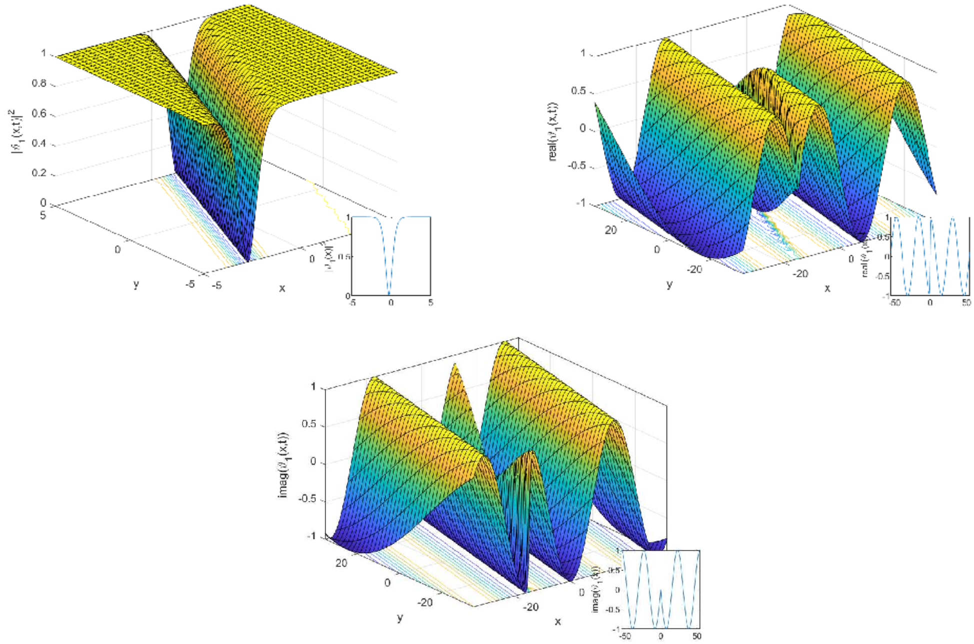

Fig. 1 – The graphics of Eq. (41) for the values of $\delta_0 = 1,\ \delta_1 = 2,\ \delta_2 = 0.3,\ \alpha = 2,\ \beta = 0.2$, and $\theta = 1$.

$$\begin{aligned}\vartheta_2(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\\ &\times \frac{\left(-\delta_1^2+4\delta_0\delta_2-\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}\coth\left(\frac{1}{2}\alpha(x-2\beta t)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\delta_1+\sqrt{\delta_1^2-4\delta_0\delta_2}\coth\left(\frac{1}{2}\alpha(x-2\beta t)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}.\end{aligned} \tag{42}$$

Plugging Eq. (13) in Eq. (38), yields

$$\begin{aligned}\vartheta_3(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\\ &\times \frac{\left(-\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}-2\mathrm{i}\delta_0\delta_2\cosh\left(\alpha(x-2\beta t)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}{\delta_1^2-2\delta_0\delta_2+2\mathrm{i}\delta_0\delta_2\sinh\left(\alpha(x-2\beta t)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)}.\end{aligned} \tag{43}$$

Plugging Eq. (14) in Eq. (38), yields

$$\begin{aligned}\vartheta_4(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\\ &\times \frac{\left(-\delta_1^2+4\delta_0\delta_2-\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}\coth\left(\frac{1}{2}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}{\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}+\left(\delta_1^2-4\delta_0\delta_2\right)\coth\left(\frac{1}{2}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)}.\end{aligned} \tag{44}$$

Plugging Eq. (15) in Eq. (38), yields

$$\begin{aligned}\vartheta_5(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\\ &\times \frac{\left(-\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}+2\delta_0\delta_2\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}{\delta_1^2-2\delta_0\delta_2-2\delta_0\delta_2\cosh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)}.\end{aligned} \tag{45}$$

Plugging Eq. (16) in Eq. (38), the singular solution is attained

$$\begin{aligned}\vartheta_6(x,t) &= -\delta_1\,\frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{\sqrt{\delta_1^2-4\delta_0\delta_2}}\\ &+ \frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{\sqrt{\delta_1^2-4\delta_0\delta_2}}\\ &\times \left(\frac{4\delta_0\delta_2\left(B+A\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right)}{\Omega+A\left(\begin{array}{c}\sqrt{\delta_1^2-4\delta_0\delta_2}\cosh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)+\\ \delta_1\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\end{array}\right)}\right),\end{aligned} \tag{46}$$

where $\Omega = B\delta_1-\sqrt{\left(A^2+B^2\right)\left(\delta_1^2-4\delta_0\delta_2\right)}$.

Plugging Eq. (17) in Eq. (38), the combined of hyperbolic functions solution is attained

$$\begin{aligned}\vartheta_7(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\\ &\times \frac{\left(\begin{array}{l} -B\delta 1^2+4B\delta_0\delta_2-\delta 1\sqrt{\left(-A^2+B^2\right)\left(\delta_1^2-4\delta_0\delta_2\right)}\\ -A\left(\delta_1^2-4\delta_0\delta_2\right)\,\cosh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\\ -A\delta_1\sqrt{\delta_1^2-4\delta_0\delta_2}\,\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\end{array}\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\begin{array}{l} B\delta_1+\sqrt{\left(-A^2+B^2\right)\left(\delta_1^2-4\delta_0\delta_2\right)}+\\ A\delta_1\cosh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\\ +A\sqrt{\delta_1^2-4\delta_0\delta_2}\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\end{array}\right)}.\end{aligned} \tag{47}$$

Plugging Eq. (18) in Eq. (38), the dark solution is attained as seen in Fig. 2:

$$\vartheta_8(x,t)=-\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\tanh\left(\frac{1}{2}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right). \tag{48}$$

Plugging Eq. (19) in Eq. (38), the hyperbolic function is attained as seen in Fig. 3:

$$\vartheta_9(x,t)=-\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\coth\left(\frac{1}{2}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right). \tag{49}$$

Plugging Eq. (20) in Eq. (38), yields

$$\begin{aligned}\vartheta_{10}(x,t) &= -\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\mathrm{sech}\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\\ &\times \left(\mathrm{i}+\sinh\left(\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\right).\end{aligned} \tag{50}$$

REMARK 1. Using Eq. (21) in Eq. (38), provided the same solutions given in Eq. (49).

Plugging Eq. (22) in Eq. (38), yields

$$\begin{aligned}\vartheta_{11}(x,t) &= -\frac{1}{2}\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\tanh\left(\frac{1}{4}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)\\ &\times \left(1+\coth\left(\frac{1}{4}\alpha\left(x-2\beta t\right)\sqrt{\delta_1^2-4\delta_0\delta_2}\right)^2\right).\end{aligned} \tag{51}$$

When $\sigma=\delta_1^2-4\delta_0\delta_2<0$ and $\delta_1\delta_2\neq 0$ (or $\delta_0\delta_2\neq 0$) and upon examining the solutions within family 2, the subsequent solutions have been derived:

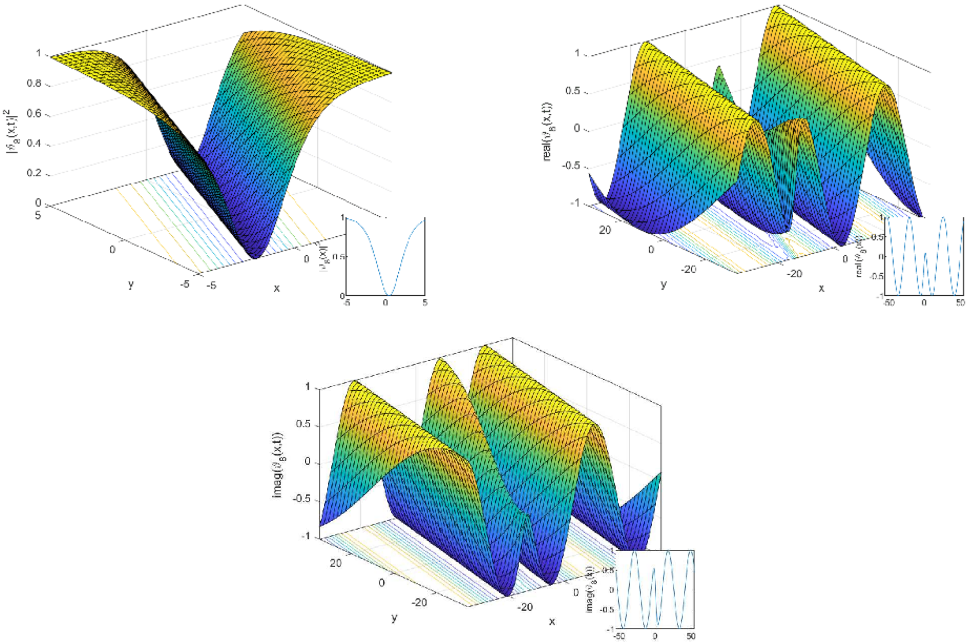

Fig. 2 – The graphics of Eq. (48) for the values of $\delta_0 = 1,\ \delta_1 = 5,\ \delta_2 = 0.3,\ \alpha = 0.2,\ \beta = 0.2$, and $\theta = 2$.

Plugging Eq. (23) in Eq. (38), yields

$$\begin{aligned}\vartheta_{12}(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)} \\ &\times \frac{\left(-\delta_1^2+4\delta_0\delta_2-\delta_1\Psi\ \tan\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\delta_1+\Psi\ \tan\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\right)},\end{aligned} \tag{52}$$

where $\Psi = \sqrt{-\delta_1^2+4\delta_0\delta_2}$. Plugging Eq. (24) in Eq. (38), yields

$$\begin{aligned}\vartheta_{13}(x,t) &= \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)} \\ &\times \frac{\left(-\delta_1^2+4\delta_0\delta_2-\delta_1\Psi\ \cot\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\delta_1+\Psi\ \cot\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\right)}.\end{aligned} \tag{53}$$

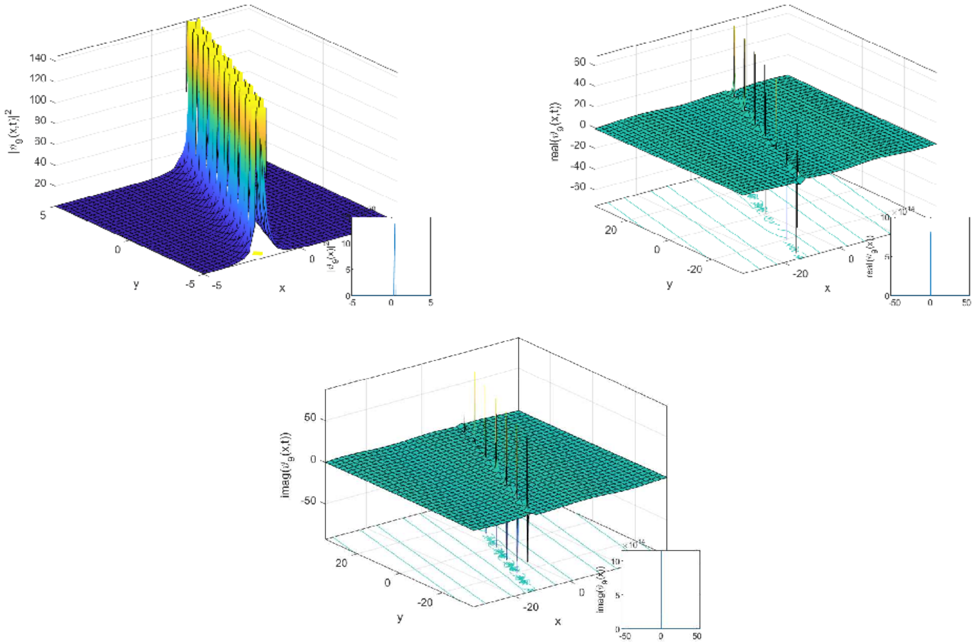

Fig. 3 – The graphics of Eq. (49) for the values of $\delta_0 = 1,\ \delta_1 = 2,\ \delta_2 = 0.3,\ \alpha = 0.2,\ \beta = 0.2$, and $\theta = 3$.

Plugging Eq. (25) in Eq. (38), yields

$$\vartheta_{14}(x,t) = \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)} \frac{\left(-\delta_1^2 + 4\delta_0\delta_2 + \delta_1\Psi\left(\begin{array}{c}\sec\left(\alpha\left(x-2t\beta\right)\Psi\right)\\ +\tan\left(\alpha\left(x-2t\beta\right)\Psi\right)\end{array}\right)\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\delta_1 - \Psi\left(\begin{array}{c}\sec\left(\alpha\left(x-2t\beta\right)\Psi\right)\\ +\tan\left(\alpha\left(x-2t\beta\right)\Psi\right)\end{array}\right)\right)}. \tag{54}$$

Plugging Eq. (26) in Eq. (38), we attain

$$\vartheta_{15}(x,t) = \mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)} \frac{\left(\begin{array}{l}-\delta_1\Psi\ \cos\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\\ -\left(\delta_1^2-4\delta_0\delta_2\right)\sin\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\end{array}\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}\left(\begin{array}{l}\Psi\ \cos\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\\ +\delta_1\sin\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\Psi\right)\end{array}\right)}. \tag{55}$$

REMARK 2. By employing Eq. (27) alongside Eq. (38), and subsequently applying the FullSimplify command in Mathematica, one arrives at the identical outcome presented in Eq. (53).

REMARK 3. By employing Eq. (28) alongside Eq. (38), and subsequently applying the FullSimplify command in Mathematica, one arrives at the identical outcome presented in Eq. (55).

Plugging Eq. (29) in Eq. (38), yields

$$\begin{aligned}\vartheta_{16}(x,t) &= \frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{\sqrt{\delta_1^2-4\delta_0\delta_2}}\\ &\times \left(\frac{4\delta_0\delta_2\left(B+A\sin\left(\alpha\left(x-2t\beta\right)\Psi\right)\right)}{\begin{array}{l}B\delta_1-\sqrt{B^2-A^2}\Psi+\\ A\ \Psi\cos\left(\alpha\left(x-2t\beta\right)\Psi\right)+A\delta_1\sin\left(\alpha\left(x-2t\beta\right)\Psi\right)\end{array}}\right)\\ &- \delta_1\frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{\sqrt{\delta_1^2-4\delta_0\delta_2}}. \qquad (56)\end{aligned}$$

Plugging Eq. (30) in Eq. (38), yields

$$\vartheta_{17}(x,t)=\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\frac{\sqrt{\delta_1^2-4\delta_0\delta_2}\tan\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\sqrt{-\delta_1^2+4\delta_0\delta_2}\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}}. \qquad (57)$$

Plugging Eq. (31) in Eq. (38), yields

$$\vartheta_{18}(x,t)=-\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}\frac{\sqrt{-\delta_1^2+4\delta_0\delta_2}\cot\left(\frac{1}{2}\alpha\left(x-2t\beta\right)\sqrt{-\delta_1^2+4\delta_0\delta_2}\right)}{\sqrt{\delta_1^2-4\delta_0\delta_2}}. \qquad (58)$$

Plugging Eq. (32) in Eq. (38), we attain

$$\vartheta_{19}(x,t)=\frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{\sqrt{\delta_1^2-4\delta_0\delta_2}}\left(\Psi\ \left(\sec\left(\alpha\left(x-2t\beta\right)\Psi\right)+\tan\left(\alpha\left(x-2t\beta\right)\Psi\right)\right)\right). \qquad (59)$$

REMARK 4. By employing Eq. (33) alongside Eq. (38), and subsequently applying the FullSimplify command in Mathematica, one arrives at the identical outcome presented in Eq. (58).

In the scenario where $\delta_0=0$ and $\delta_1\delta_2\neq 0$, and considering the solutions within family 3, the subsequent solution is obtained:

Plugging Eq. (34) in Eq. (38), we attain

$$\vartheta_{20}(x,t) = -\frac{\mathrm{e}^{\mathrm{i}\left(-\beta x+\beta^2 t+\theta_0\right)}}{2\sqrt{\delta_1^2-4\delta_0\delta_2}} \left(\begin{array}{l} \Psi \left(-1+\cot\left(\dfrac{1}{4}\alpha\left(x-2t\beta\right)\Psi\right)^2\right) \\ \times\tan\left(\dfrac{1}{4}\alpha\left(x-2t\beta\right)\Psi\right) \end{array} \right). \tag{60}$$

REMARK 5. By employing Eq. (35) in conjunction with Eq. (38), and subsequently applying the FullSimplify command in Mathematica, one arrives at a result that is consistent with that presented in Eq. (60).

It is important to note that the expression Ψ, which appears in (53), (54), (55), (56), (59), and (60), is the same as the one introduced following (52).

6. CONCLUSION

In this paper, we have carried out a comprehensive symmetry-based analysis of a nonlinear coupled evolution system involving the dependent variables $P(x,t)$ and $Q(x,t)$. The admitted Lie point symmetries were determined and shown to form a Lie algebra generated by translations in space and time together with a scaling transformation. The structure of this algebra was clarified through the construction of the commutator table and adjoint representation, and its classification was achieved by deriving an optimal system of subalgebras. These results enabled systematic reductions of the governing PDE system into ordinary differential equations (ODEs).

The reduced ODEs were further analyzed using power series expansions, thereby providing approximate analytic solutions and illustrating the effectiveness of the reduction procedure. In addition, by employing a new conservation theorem, conservation laws associated with each symmetry generator were derived. These conservation laws enrich the structural understanding of the system, as they are directly connected to physically relevant invariants such as energy and momentum.

Overall, the combination of Lie symmetry methods and Ibragimov's conservation law framework has provided a unified approach to uncovering both the algebraic structure and the invariant properties of the system. The findings not only contribute to the theoretical development of symmetry analysis in nonlinear PDEs but also pave the way for future research on integrability, exact solution construction, and the design of numerical schemes that preserve conservation laws.

FUTURE WORK

Future studies may extend the present framework in several directions. One natural extension is to investigate the fractional generalization of the system using

Caputo-type derivatives, which would provide deeper insights into memory effects and anomalous diffusion phenomena. Another promising direction is the exploration of potential integrability of the system through the Painlevé test and the construction of Lax pairs. Moreover, numerical simulations that preserve the derived conservation laws could be designed and compared with classical schemes to highlight their advantages in long-time integration. Finally, the methodology developed here can be applied to broader classes of nonlinear coupled PDEs arising in fluid dynamics, nonlinear optics, and plasma physics, thereby enhancing its relevance to both mathematics and applied sciences.

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